

CRITICAL PERCOLATION AND THE MINIMAL SPANNING TREE IN SLABS

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ABSTRACT. The minimal spanning forest on \mathbb{Z}^d is known to consist of a single tree for $d \leq 2$ and is conjectured to consist of infinitely many trees for large d . In this paper, we prove that there is a single tree for quasi-planar graphs such as $\mathbb{Z}^2 \times \{0, \dots, k\}^{d-2}$. Our method relies on generalizations of the “Gluing Lemma” of [DST15a]. A related result is that critical Bernoulli percolation on a slab satisfies the box-crossing property. Its proof is based on a new Russo-Seymour-Welsh type theorem for quasi-planar graphs. Thus, at criticality, the probability of an open path from 0 of diameter n decays polynomially in n . This strengthens the result of [DST15a], where the absence of an infinite cluster at criticality was first established.

1. INTRODUCTION

There are two standard models of random spanning trees on finite graphs: the *uniform* spanning tree and the *minimal* spanning tree. One can define these, by taking a limit, on infinite graphs, such as \mathbb{Z}^d with nearest-neighbor edges, but then the single finite spanning tree may become a forest of many disjoint trees. Because the uniform spanning tree is closely related to random walks and potential theory ([Wil96], see also [BLPS01]), it is known [Pem91] that the critical dimension is exactly $d_c = 4$, only above which is there more than a single tree.

In the case of the minimal spanning tree, where random walks and potential theory are replaced by invasion percolation and critical Bernoulli percolation, very little is known rigorously. As we will discuss in more detail below, it is known that there is a single tree in \mathbb{Z}^2 ([CCN85], see also [AM94]) and there are conjectures that for large d there are infinitely many trees. The main purpose of this paper is to make progress toward a proof that at least for some low dimensions above $d = 2$, there is a single tree by showing that this is the case for the approximation of, say, \mathbb{Z}^3 by a thick slab $\mathbb{Z}^2 \times \{0, \dots, k\}$. In the process, we also obtain a new result for critical percolation on such slabs, where it was only recently proved that there is no infinite cluster [DST15a]; the new result is inverse power law decay for the probability of large diameter finite clusters at criticality (see Corollary 3.2 in Section 3).

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Bibliographic note: After the current paper was finished, we learned that an alternate proof of the lower bound in the box-crossing property of Theorem 3.1 (existence of open crossings in long rectangles with positive probability) was obtained independently in the very recent paper [BS15], using a different argument.

To define the minimal spanning tree (MST) on a finite connected graph $G = (V, E)$, assign random weights $\{\omega(e) : e \in E\}$, that are i.i.d. uniform $[0, 1]$ random variables, to its edges. The MST is the spanning tree that minimizes the total weight. Equivalently, it can be obtained from G by deleting every edge whose weight is maximal in some cycle. When G is an infinite graph, two natural infinite volume limits can be taken, which lead to the notion of free and wired minimal spanning forests. See [Ale95], [Häg95] and [LPS06] for basic properties of minimal spanning forests on infinite graphs.

On \mathbb{Z}^d , it is known that the free and wired minimal spanning forests coincide (see Proposition 2.2). Therefore, in this framework, we can talk about *the* minimal spanning forest without ambiguity. Although it arises as the weak limit of minimal spanning trees on finite graphs, the minimal spanning forest may no longer be a single tree, and can even have infinitely many components. A natural question is, for which d is the minimal spanning forest in \mathbb{Z}^d almost surely a single tree? This question is largely open except for $d = 2$ (and trivially for $d = 1$), where the minimal spanning forest is known to be a single tree ([CCN85], [AM94]); the argument there crucially relies on the planarity, and does not apply to $d \geq 3$. A much more modest question, whether the number of components in minimal spanning forest in \mathbb{Z}^d is either 1 or ∞ almost surely, also remains open. Besides its own interest, the number of connected components of minimal spanning forests is also closely related to the ground state structure of nearest-neighbor spin glasses and other disordered Ising models [NS96]. It is believed that there is a finite upper critical dimension d_c , below which the minimal spanning forest is a single tree a.s., and above which it has infinitely many components a.s. Based on a combination of rigorous and heuristic arguments, there have been interesting competing conjectures that $d_c = 8$ [NS96] (see also [NS94]) or $d_c = 6$ [Jac10]. But it has not even been proved that there are multiple trees for very large d .

Another natural random forest measure on infinite graphs is the uniform spanning forest, defined as the weak limit of uniform spanning trees on finite subgraphs. As mentioned earlier, the geometry of uniform spanning forests is much better understood, because of the connections to random walks and potential theory ([Wil96], see also [BLPS01]). Its upper critical dimension is thus closely related to the intersection probability of random walks. It was shown in [Pem91] that here $d_c = 4$.

Minimal spanning forests are closely related to critical Bernoulli percolation and invasion percolation. Just as (wired) uniform spanning forests can be constructed by piecing together loop erased random walks by Wilson's algorithm [Wil96], (wired) minimal spanning forests can be constructed using invasion trees (see Proposition 2.3 below). On the two dimensional triangular lattice, Garban, Pete and Schramm [GPS14] proved that the minimal spanning tree on this graph has a scaling limit, based on fine knowledge of near-critical Bernoulli percolation.

In this paper, we study the minimal spanning forests on a class of non-planar infinite graphs, namely two dimensional slabs, whose vertex set is of the form $\mathbb{Z}^2 \times \{0, \dots, k\}^{d-2}$, for $k \in \mathbb{N}$. Although $d > 2$ can be arbitrary here, these graphs are all quasi-planar.

A main result of this paper is Theorem 2.4, which states that on any two dimensional slab, the minimal spanning forest is a single tree a.s. The argument also applies to other quasi-planar graphs, such as \mathbb{Z}^2 with non-nearest neighbor edges up to a finite distance — see the remark after Theorem 2.4.

An important ingredient in the proof is the box-crossing property for critical Bernoulli percolation on slabs, stated as Theorem 3.1. Its proof is based on a Russo-Seymour-Welsh type theorem, and extends to a larger class of models — e.g., Bernoulli percolation on quasi-planar graphs invariant under a non trivial rotation, or short-range Bernoulli percolation on \mathbb{Z}^2 invariant under $\pi/2$ -rotations.

Because of the relation between the minimal spanning forest and critical Bernoulli percolation, it is not surprising that we adapt tools from the percolation literature. Indeed, a major open question in Bernoulli percolation is to prove in \mathbb{Z}^d , $3 \leq d \leq 6$, that there is no percolation at the critical point. Although this question is still beyond reach, it was recently proved in [DST15a] that non-percolation at criticality is valid for two-dimensional slabs. A key technical ingredient in that proof is a gluing lemma for open paths (see Theorem 3.7 below for a more general version). In this paper we use a related gluing lemma (Lemma 4.1) that applies to invasion trees and minimal spanning trees.

This paper is organized as follows. In Section 2 we collect definitions and basic properties for minimal spanning forests and invasion percolation, describe their connections, state the main result (Theorem 2.4) for minimal spanning forests on slabs and sketch our proof. Section 3 is devoted to the proof of Russo-Seymour-Welsh type box-crossing theorems on slabs, which are used in our argument and are also of interest in their own right (see especially Theorem 3.1 and Corollary 3.2). Finally, in Section 4, we collect all the ingredients to prove our gluing lemma for invasion clusters, and thus conclude the proof of Theorem 2.4. We note that one ingredient, Lemma 4.2, is an extension of the combinatorial lemma of [DST15a]. The extension is needed for the invasion setting where continuous edge variables replace Bernoulli ones.

2. BACKGROUND AND FIRST MAIN RESULT

2.1. Minimal Spanning Forests. Let $G = (V, E)$ be a finite graph. A subgraph H of G is spanning if H contains all vertices of G . A labeling is an injective function $\omega : E \rightarrow [0, 1]$. The number $\omega_e \doteq \omega(e)$ will be referred to as the label of e . Note that the labeling induces a total ordering on E , where $e \prec e'$ if $\omega(e) < \omega(e')$.

Define \mathcal{T}^ω to be a spanning subgraph of G whose edge set consists of all $e \in E$ whose endpoints cannot be joined by a path whose edges are all strictly smaller than e . It is easy to see that \mathcal{T}^ω is a spanning tree, and in fact, among all spanning trees \mathcal{T} , \mathcal{T}^ω minimizes $\sum_{e \in \mathcal{T}} \omega(e)$ ([LPS06]).

Definition 2.1. When $\{\omega(e) : e \in E\}$ are i.i.d. uniform $[0, 1]$ random variables, the law of the corresponding spanning tree \mathcal{T}^ω is called the minimal spanning tree (MST). The law of \mathcal{T}^ω defines a probability measure on 2^E (where we identify the tree \mathcal{T}^ω with its set of edges).

When passing to infinite graphs, two natural definitions of minimal spanning forests can be made, that arise as weak limits of minimal spanning trees on finite graphs.

Let $G = (V, E)$ be an infinite graph, and $\omega : E \rightarrow [0, 1]$ be a labeling function. Let \mathcal{F}_f^ω be the set of edges $e \in E$, such that in every path in G connecting the endpoints of e , there is at least one edge e' with $\omega(e') > \omega(e)$. When $\{\omega(e) : e \in E\}$ are i.i.d. uniform $[0, 1]$ random variables, the law of \mathcal{F}_f^ω is called the free minimal spanning forest (FMSF) on G .

An extended path joining two vertices $a, b \in V$ is either a simple path in G joining them, or the disjoint union of two simple semi-infinite paths, starting at a, b respectively. Let \mathcal{F}_w^ω be the set of edges $e \in E$, such that in every extended path in G connecting the endpoints of e , there is at least one edge e' with $\omega(e') > \omega(e)$. Analogously, when $\{\omega(e) : e \in E\}$ are i.i.d. uniform $[0, 1]$ random variables, the law of \mathcal{F}_w^ω is called the wired minimal spanning forest (WMSF) on G .

It is clear that \mathcal{F}_f^ω and \mathcal{F}_w^ω are indeed forests. In addition, all the connected components in \mathcal{F}_f^ω and \mathcal{F}_w^ω are infinite. In fact, the smallest label edge joining any finite vertex set to its complement belongs to both forests.

We now describe how \mathcal{F}_f^ω and \mathcal{F}_w^ω arise as weak limits of minimal spanning trees on finite graphs. Consider an increasing sequence of finite, connected induced subgraphs $G_n \subset G$, such that $\cup_{n \geq 1} G_n = G$. For $n \in \mathbb{N}$, let G_n^w be the graph obtained from G by identifying the vertices in $G \setminus G_n$ to a single vertex.

Proposition 2.1 ([Ale95][LPS06]). *Let $\mathcal{T}_n^\omega, \overline{\mathcal{T}}_n^\omega$ denote the minimal spanning tree on G_n and G_n^w , respectively, that are induced by the labeling ω . Then for any labeling function ω ,*

$$\mathcal{F}_f^\omega = \lim_{n \rightarrow \infty} \mathcal{T}_n^\omega, \text{ and } \mathcal{F}_w^\omega = \lim_{n \rightarrow \infty} \overline{\mathcal{T}}_n^\omega.$$

This means for every $e \in \mathcal{F}_f^\omega$, we have $e \in \mathcal{T}_n^\omega$ for all sufficiently large n , and similarly for \mathcal{F}_w^ω .

One natural question on a given connected graph is whether the free and wired minimal spanning forests coincide. To answer this question, we need to explain the relation to critical Bernoulli percolation.

Proposition 2.2 ([LPS06],[Ale95]). *On any connected graph G , we have $\mathcal{F}_f^\omega = \mathcal{F}_w^\omega$ if and only if for almost every $p \in (0, 1)$, Bernoulli percolation on G with parameter p has at most one infinite cluster a.s.*

Let $[k] := \{0, \dots, k\}$ and $\mathbb{S}_k := \mathbb{Z}^2 \times [k]$ be the slab of thickness k . It follows from [AKN87] and [BK89] that the infinite cluster on \mathbb{S}_k , if it exists, is a.s. unique. Therefore on \mathbb{S}_k , WMSF and FMSF coincide. This justifies referring to *the* minimal spanning forest on \mathbb{S}_k without ambiguity.

2.2. Invasion Percolation. We now define invasion percolation, an object closely related to WMSF and critical Bernoulli percolation. Let $\{\omega(e) : e \in E\}$ be i.i.d. uniform $[0, 1]$ random variables. The invasion cluster \mathcal{I}_v of a vertex v is defined as a union of subgraphs

$\mathcal{I}_v(k)$, where $\mathcal{I}_v(0) = \{v\}$, and $\mathcal{I}_v(k+1)$ is $\mathcal{I}_v(k)$ together with the lowest labeled edge (and its vertices) not in $\mathcal{I}_v(k)$ but incident to some vertex in $\mathcal{I}_v(k)$.

We also define the invasion tree, T_v of a vertex v , as the increasing union of trees $T_v(k)$, where $T_v(0) = \{v\}$, and $T_v(k+1)$ is $T_v(k)$ together with the lowest edge (and its vertices) joining $T_v(k)$ to a vertex *not* in $T_v(k)$. Notice that \mathcal{I}_v has the same vertices as T_v , but may have additional edges.

The following proposition in [LPS06] (see also [NS96]) describes the relation between invasion trees and WMSF.

Proposition 2.3. *Let $G = (V, E)$ be a locally finite graph. Then*

$$\mathcal{F}_w^\omega = \cup_{v \in V} T_v \quad a.s.$$

Therefore, to show \mathcal{F}_w^ω is a single tree, it suffices to prove for any $v \in V$, $\mathcal{I}_0 \cap \mathcal{I}_v \neq \emptyset$.

We now describe the connection between invasion percolation and critical Bernoulli percolation. An edge $e \in E$ is said to be p -open if its weight satisfies $\omega(e) < p$. The connected components of the graph induced by the p -open edges are called p -open clusters. Notice that the set of p -open edges is a Bernoulli bond percolation process on G with edge density p .

Let $p_c(G)$ be the critical probability for Bernoulli bond percolation on G . For any $p > p_c(G)$, there exists almost surely an infinite p -open cluster. Suppose that for some k , $\mathcal{I}_v(k)$ contains a vertex of this cluster. Then all edges invaded after time k remain in this cluster.

To make another observation, denote by $\mathcal{C}_{p_c}(v)$ the p_c -open cluster of a vertex $v \in G$, and write $\theta_v(p_c)$ for the probability that $\mathcal{C}_{p_c}(v)$ is infinite. If $\theta_v(p_c) = 0$, then of course the p_c -open cluster $\mathcal{C}_{p_c}(v)$ is finite a.s. This implies that once v is reached by an invasion, then (with probability 1) all edges in $\mathcal{C}_{p_c}(v)$ will be invaded before any edges with label $\geq p_c$ are invaded. In particular, when $\theta_v(p_c) = 0$, the p_c -cluster of v satisfies $\mathcal{C}_{p_c}(v) \subset \mathcal{I}_v$.

2.3. Notation, conventions. We consider the space of configurations $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = [0, 1]^E$ (E denotes the edge set of the slab \mathbb{S}_k , given by the pairs of points at Euclidean distance 1 from each other), \mathcal{F} is the Borel σ -field on Ω , and \mathbb{P} is the underlying (product of uniforms) probability measure. Given the labelling function $\omega \in \Omega$, and $S \subset E$, we use $\omega|_S$ to denote the restriction of ω to S .

Given $a, b \in \mathbb{Z}$, $a < b$, let $[a, b] = \{a, a+1, \dots, b\}$, and we simply denote by $[k]$ the set $[0, k]$. For any subset $S \subset \mathbb{Z}^2$, we denote by \overline{S} the set $S \times [k] \subset \mathbb{S}_k$, for $z \in \mathbb{Z}^2$, we denote by \overline{z} the set $\{z\} \times [k]$, and for $S \subset \mathbb{S}_k$, we denote by \overline{S} the set $\pi(S) \times [k]$, where $\pi(S)$ is the projection of S onto \mathbb{Z}^2 . For $x \in \mathbb{S}_k$ and $U, V \subset \mathbb{S}_k$, we denote by $|x|$ the Euclidean norm of x , and $\text{dist}(U, V) = \min_{u \in U, v \in V} |u - v|$. For $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{S}_k$ and $U, V \subset \mathbb{S}_k$, we define $\text{dist}^*(x, y) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$, and $\text{dist}^*(U, V) = \min_{x \in U, y \in V} \text{dist}^*(x, y)$. The vertex-boundary of a set $U \subset \mathbb{S}_k$ is denoted by ∂U (it is defined as the set of vertices in U with a neighbor in $\mathbb{S}_k \setminus U$). Given $m > n > 0$, define $B_n(x) = x + [-n, n]^2$ and $A_{n,m}(x) = B_m(x) \setminus B_n(x)$. When x is the origin, we will

omit the dependence on x . If $z \in \mathbb{S}_k$, we write $B_n(z)$ for the ball $B_n(\pi(z))$, where $\pi(z)$ is the projection of z onto \mathbb{Z}^2 .

2.4. Single Tree Result.

Theorem 2.4. *For any $k \in \mathbb{N}$, the minimal spanning forest on \mathbb{S}_k is a single tree a.s.*

Remark. As we will see from the proof below, the same argument applies to $\mathbb{Z}^2 \times F$, where F is any finite connected graph. This includes $F = \{0, \dots, k\}^{d-2}$, for $d \geq 3$. Similar arguments also apply to the finite range extensions $\mathbb{Z}_K^2 = (\mathbb{Z}^2, E_K)$ of \mathbb{Z}^2 , where $E_K = \{(x, y) : |x - y| \leq K\}$.

A sketch of the proof is as follows; the complete proof is in Sections 3 and 4 below. By Proposition 2.3, it suffices to prove that for any $x \in \mathbb{S}_k$, $\mathcal{I}_0 \cap \mathcal{I}_x \neq \emptyset$. This is shown in two steps.

1. We first prove that with bounded away from zero probability, there is a p_c -open circuit in the annulus $\overline{A_{n,2n}}$. This follows from the box-crossing property for critical Bernoulli percolation on \mathbb{S}_k which we will prove in Section 3 — see Theorem 3.1. The proof uses a new Russo-Seymour-Welsh type theorem, based on gluing lemmas given in Section 3 below (like those first established in [DST15a]).

2. It follows from Step 1 that infinitely many disjoint p_c -open circuits in \mathbb{S}_k “surround” the origin. In particular, the projections of \mathcal{I}_0 and \mathcal{I}_x on \mathbb{Z}^2 intersect the projections of p_c -open circuits infinitely many times. In Section 4, we prove a version of the gluing lemma adapted to invasion clusters, which says, roughly speaking, that each time the invasion cluster \mathcal{I}_0 crosses an annulus $\overline{A_{n,2n}}$, it “glues” to a p_c -open circuit \mathcal{C}_n in this annulus with probability larger than a constant $c > 0$ (independent of n). The same argument applies to the invasion cluster \mathcal{I}_x . Therefore, with probability 1, \mathcal{I}_0 and \mathcal{I}_x eventually are glued to the same p_c -open circuit, which implies that $\mathcal{I}_0 \cap \mathcal{I}_x \neq \emptyset$.

3. RSW THEORY AND POWER LAW DECAY ON SLABS

In this section, we consider Bernoulli bond percolation with density p on the slab \mathbb{S}_k : each edge is declared independently open with probability p and closed otherwise. Write \mathbf{P}_p for the resulting probability measure on the configuration space $\{0, 1\}^E$. Let $R = \overline{[x, x'] \times [y, y']}$ be a rectangle in \mathbb{S}_k . We say that R is crossed horizontally if there exists an open path from $\overline{\{x\} \times [y, y]}$ to $\overline{\{x'\} \times [y, y]}$ inside R . We denote this event by $\mathcal{H}(R)$. For $m, n \geq 1$, we define for $p \in [0, 1]$,

$$f(m, n) = f_p(m, n) := \mathbf{P}_p \left[\mathcal{H} \left(\overline{[0, m] \times [0, n]} \right) \right].$$

In this section, we prove the following result, that the box-crossing property holds for critical Bernoulli percolation on the slab \mathbb{S}_k , for every fixed $k \geq 1$.

Theorem 3.1 (Box-crossing property). *Let $p = p_c(\mathbb{S}_k)$. For every $\rho > 0$, there exists a constant c_ρ such that for every $n \geq 1/\rho$,*

$$c_\rho \leq f(n, \lfloor \rho n \rfloor) \leq 1 - c_\rho.$$

Remark. In our proof, the bound c_ρ we obtain depends on the thickness k of the slab. Due to our use of the gluing lemma, the bounds we obtain get worse when the thickness of the slab increases. More precisely, for fixed $\rho > 0$ and for the slab with thickness k , our proof provides us with a constant $c_\rho = c_\rho(k)$, and the sequence $(c_\rho(k))$ converges quickly to 0 as the thickness k tends to infinity. Getting better bounds would be very interesting to help understand critical behavior on \mathbb{Z}^3 (which corresponds to $k = \infty$).

The box-crossing property was established by Kesten (see [Kes82]) for critical Bernoulli percolation on two dimensional lattices under a symmetry assumption. The proof relies on a result of Russo, Seymour and Welsh ([Rus78, SW78]) which relates crossing probabilities for rectangles with different aspect ratios. Recently, the box-crossing property has been extended to planar percolation processes with spatial dependencies, e.g. continuum percolation [Tas15, ATT] or the random-cluster model [DST15b].

The box-crossing property has been instrumental in many works on Bernoulli percolation, and has numerous applications. These include Kesten's scaling relations [Kes87], bounds on critical exponents (e.g. polynomial bounds on the one arm event), computation of universal exponents and tightness arguments in the study of the scaling limit [Smi01], to name a few. We expect that similar results can be derived from the box-crossing property of Theorem 3.1. Next, we state some direct useful consequences of the box-crossing property (these are proved in Section 3.8).

Corollary 3.2. *For critical Bernoulli percolation on the slab \mathbb{S}_k , we have:*

1. [Existence of circuits with positive probability] *There exists $c > 0$ such that for every $n \geq 1$,*

$$\mathbf{P}_p[\text{there exists an open circuit in } \overline{A_{n,2n}} \text{ surrounding } \overline{B_n}] \geq c.$$

2. [Existence of blocking surfaces with positive probability] *There exists $c > 0$ such that for every $n \geq 1$,*

$$\mathbf{P}_p[\text{there exists an open path from } \overline{B_n} \text{ to } \partial \overline{B_{2n}}] \leq 1 - c.$$

3. [Polynomial decay of the 1-arm event] *There exists $\delta > 0$, such that for $n > m \geq 1$,*

$$\mathbf{P}_p[\text{there exists an open path from } \overline{B_m} \text{ to } \partial \overline{B_n}] \leq (m/n)^\delta.$$

Remarks.

1. Item 2 can be interpreted geometrically via duality: there is no open path from $\overline{B_n}$ to $\partial \overline{B_{2n}}$ if and only if there exists a blocking closed surface in the annulus $\overline{A_{n,2n}}$, made up of the plaquettes in the dual lattice perpendicular to the closed edges in \mathbb{S}_k . Therefore,

Item 2 can be understood as the existence with probability at least c of such a blocking surface in the annulus $\overline{A_{n,2n}}$.

2. Item 3 implies in particular that critical percolation on the slab \mathbb{S}_k does not have an infinite cluster. It strengthens the previous result of [DST15a]. Moreover, our proof leads to the bound $\delta \geq C^{-k}$ for some $C < \infty$. Strengthening the lower bound of $\delta(k)$ would also be very interesting.

Planar geometry is a key ingredient for the proofs of the existing box-crossing results on planar graphs. In the case of non-planar graphs, one side of the inequality can still be proved by standard renormalization arguments. Namely, the crossing probability of the short side of the rectangle is bounded from below. This is sketched as Lemma 3.12 in Section 3.4. The more difficult part is to carry out a renormalization argument to prove the crossing probability of the long side is bounded from above (this is done in Lemma 3.11), and to relate the crossing probability of the long side to that of the short side (done in Section 3.5). This is where the quasi-planarity of the slabs comes into play. For planar graphs such relations can be obtained by repeated use of the Harris-FKG inequality and one of its consequence known as the square root trick. For non-planar graphs, two paths may not intersect even if their projections on the plane do intersect. In Sections 3.2 we prove versions of gluing lemmas for open paths and circuits in \mathbb{S}_k . In Section 3.3 we apply the gluing lemmas to bound the crossing probability for rectangles with different aspect ratio. Finally, we put these ingredients together and complete the proof of RSW type theorems in Section 3.5, 3.6 and 3.7.

3.1. Positive correlation and the square-root trick. In this section we recall the Harris-FKG inequality about positive correlation of increasing events, and an important consequence called the square-root trick. We refer to [Gri99] for more details. A percolation event $\mathcal{A} \subset \{0,1\}^E$ is said to be increasing if

$$\left. \begin{array}{l} \omega \in \mathcal{A} \\ \forall e \in E, \omega(e) \leq \eta(e) \end{array} \right\} \implies \eta \in \mathcal{A}.$$

It is decreasing if \mathcal{A}^c is increasing.

Theorem 3.3 (Harris-FKG inequality). *Let $p \in [0, 1]$. Let \mathcal{A}, \mathcal{B} be two increasing events (or two decreasing events), then*

$$\mathbf{P}_p[\mathcal{A} \cap \mathcal{B}] \geq \mathbf{P}_p[\mathcal{A}] \mathbf{P}_p[\mathcal{B}].$$

The following straightforward consequence, called the square-root trick, will be very useful.

Corollary 3.4 (Square-root trick). *Let $\mathcal{A}_1, \dots, \mathcal{A}_j$ be j increasing events, then*

$$\max\{\mathbf{P}_p[\mathcal{A}_1], \dots, \mathbf{P}_p[\mathcal{A}_j]\} \geq 1 - (1 - \mathbf{P}_p[\mathcal{A}_1 \cup \dots \cup \mathcal{A}_j])^{1/j}.$$

3.2. Gluing Lemmas. Define \mathcal{H} to be the set of continuous and strictly increasing functions $h : [0, 1] \rightarrow [0, 1]$. Clearly, given $h_1, h_2 \in \mathcal{H}$, we have $h_1 h_2 \in \mathcal{H}$, $h_1^{-1} \in \mathcal{H}$, and $1 - h_1^{-1}(1 - \cdot) \in \mathcal{H}$. We sometimes denote by h a function in \mathcal{H} that may change from line to line.

In order to state the gluing lemmas we need to fix an ordering \prec on the vertices of \mathbb{S}_k . The choice of the ordering is flexible; ours is the following. Given $x, y \in \mathbb{S}_k$, we write $x \prec y$ iff

- $|x| < |y|$, or
- $|x| = |y|$, and there exists k such that $x_i = y_i$ for $i < k$, and $x_k < y_k$.

We order directed edges and more generally, site self-avoiding paths of \mathbb{S}_k by taking the corresponding lexicographical order, as in Section 2.3 of [DST15a]. Let S be a connected subset of \mathbb{S}_k . For $A, B, S \subset \mathbb{S}_k$, the event $A \xrightarrow{S} B$ denotes the existence of a path of open edges in S connecting $A \cap S$ to $B \cap S$. If this event occurs, define $\Gamma_{\min}^S(A, B)$ to be the minimal (for the order defined above) open self-avoiding path in S from A to B . Set $\Gamma_{\min}^S(A, B) = \emptyset$ if there is no open path from A to B in S . Note that $\Gamma_{\min}^S(A, B)$ is defined relative to a fixed S . Sometimes we will also use the definitions above with A and B random sets (they may depend on the configuration ω).

We will repeatedly use the following combinatorial lemma stated in [DST15a].

Lemma 3.5. *Let $s, t > 0$. Consider two events \mathcal{A} and \mathcal{B} and a map Φ from \mathcal{A} into the set $\mathfrak{P}(\mathcal{B})$ of subevents of \mathcal{B} . We assume that:*

- (1) *for all $\omega \in \mathcal{A}$, $|\Phi(\omega)| \geq t$,*
- (2) *for all $\omega' \in \mathcal{B}$, there exists a set S with less than s edges such that $\{\omega : \omega' \in \Phi(\omega)\} \subset \{\omega : \omega|_S = \omega'|_S\}$.*

Then,

$$\mathbf{P}_p[\mathcal{A}] \leq \frac{1}{t} \left(\frac{2}{\min\{p, 1-p\}} \right)^s \mathbf{P}[\mathcal{B}].$$

We now state two gluing lemmas for open paths crossing subsets of rectangular regions of the form $R \times [k]$, with R a topological rectangle. The first, Theorem 3.6, has the simplest geometry and will be proved as a consequence of essentially the same arguments used for Theorem 3.7, which has a more complicated geometry.

As in [DST15a], the proof of these gluing lemmas uses local modifications of percolation configurations, which rely on the following definition.

Definition 3.1. Define an integer $r \geq 3$ such that for every $s \geq r$ and every $z \in L \doteq \mathbb{Z}_+^2 \times [k] \setminus \{(0, 0, 0), (0, 0, k)\}$, the following holds. For any three distinct neighbors u, v, w of z , and any three distinct sites u', v', w' (that are also distinct from u, v, w) on the boundary of $\overline{B_s(z)} \cap L$, there exist three disjoint self-avoiding paths in $\overline{B_s(z)} \cap L \setminus \{z\}$ connecting u to u' , v to v' and w to w' .

In the case of slabs with $k \geq 1$, it suffices to take $r = 3$. We will present the proof with general r , since that can be adapted to the more general quasi-planar graphs $\mathbb{Z}^2 \times F$, with F a finite connected graph.

Theorem 3.6. *Let $r \geq 3$ be as in Definition 3.1. Fix $\varepsilon > 0$ and $k \geq 1$. There exists $\mathbf{h}_0 \in \mathcal{H}$ such that the following holds. Let S be a subset of \mathbb{S}_k of the form $\overline{[a, b] \times [c, d]}$, with $b - a \geq r + 2$, $d - c \geq r + 2$. Let A, B, C, D be four subsets of ∂S such that their projections on \mathbb{Z}^2 are disjoint, and such that the projection on \mathbb{Z}^2 of any path from A to B in S intersects the projection of any path from C to D in S . Then for every $p \in [\varepsilon, 1 - \varepsilon]$,*

$$\mathbf{P}_p \left[C \xleftrightarrow{S} A \right] \geq \mathbf{h}_0(\mathbf{P}_p \left[A \xleftrightarrow{S} B \right] \wedge \mathbf{P}_p \left[C \xleftrightarrow{S} D \right]). \quad (1)$$

Roughly speaking, when both $A \xleftrightarrow{S} B$ and $C \xleftrightarrow{S} D$ occur with uniformly positive probability, so does $C \xleftrightarrow{S} A$. If both $A \xleftrightarrow{S} B$ and $C \xleftrightarrow{S} D$ occur with high probability, then so does $C \xleftrightarrow{S} A$.

The proof of Theorem 3.6 (see below) is a slightly modified version of the proof of the following theorem.

Theorem 3.7 (Main gluing lemma for paths). *Let $r \geq 3$ be as in Definition 3.1. Fix $\varepsilon > 0$ and $k \geq 1$. There exists $\mathbf{h}_0 \in \mathcal{H}$ such that the following holds. Let S, R be two subsets of \mathbb{S}_k of the form $\overline{[a, b] \times [c, d]}$, with $b - a \geq r + 2$, $d - c \geq r + 2$. Let $A, B \subset \partial S$ and $C \subset \partial R$ be such that the projections of A, B, C on \mathbb{Z}^2 are at least sup-norm distance $r + 2$ apart from each other. Then for every $p \in [\varepsilon, 1 - \varepsilon]$,*

$$\mathbf{P}_p \left[C \xleftrightarrow{R} A \right] \geq \mathbf{h}_0(\mathbf{P}_p \left[C \xleftrightarrow{R} \mathcal{N}(\bar{\Gamma}, r) \right]), \quad (2)$$

where $\Gamma = \Gamma_{\min}^S(A, B)$ and $\mathcal{N}(\bar{\Gamma}, r) = \{x \in S : \text{dist}^*(x, \Gamma) \leq r\}$.

Remarks.

1. We note that in the simpler case of the plane (when $k = 0$), the FKG inequality implies that the left hand side of (1) is greater than or equal to

$$\mathbf{P}_p \left[A \xleftrightarrow{S} B \right] \mathbf{P}_p \left[C \xleftrightarrow{S} D \right],$$

and thus (1) is valid with $\mathbf{h}_0(x) = x^2$.

2. For better readability of the proof we have not presented Theorem 3.7 in the highest level of generality. In particular the sets R and S in the statements of Theorem 3.6 and Theorem 3.7 do not need to be rectangles. The proof also applies if the sets R and S are of the form \bar{T} , where $T \subset \mathbb{Z}^2$ is a rectilinear domain such that ∂T is a simple circuit made of vertical/horizontal segments of length at least $r + 2$, and such that any two disjoint segments are at least sup-norm distance $r + 2$ apart from each other.
3. In the proof we will see that the function $\mathbf{h}_0 \in \mathcal{H}$ can be chosen in such a way that $\mathbf{h}_0(x) \geq c_0 x$, where c_0 is a constant that depends only on ε and k . This remark will be important in the proof of Theorem 3.10.

Proof of Theorem 3.7. The proof consists of two parts. In the first part, we prove that for some $\delta > 0$, there is an $h_1 : [0, 1] \rightarrow [0, 1]$ that is continuous on $[0, 1]$, strictly increasing on $[1 - \delta, 1]$, and with $h_1(1) = 1$, such that (2) is valid with \mathbf{h}_0 replaced by h_1 . In the second part, we show that (2) is valid with \mathbf{h}_0 replaced by $h_2(x) = c_0 x$, $x \in [0, 1]$, for some $c_0 > 0$ (which depends only on ε and k). Therefore one can take $\mathbf{h}_0(x) = \max\{h_1(x), c_0 x\}$,

which is indeed in \mathcal{H} . The proof of the first part is very similar to the argument in Section 2.3 of [DST15a], and we next outline those arguments, with details supplied to show that h_1 is continuous and strictly increasing.

For the proof below we will slightly abuse notation and use $\overline{B_s(z)}$ to denote $\overline{B_s(z)} \cap S$.

Following Section 2.3 of [DST15a], we define $U(\omega)$, $\omega \in [0, 1]^{\mathbb{S}_k}$ to be the set of vertices $z \in S$ such that

- $z \in \overline{\Gamma}$, and
- $\overline{B_{r+1}(z)}$ is connected to C in R by an open path π , such that $\text{dist}^*(\pi, \overline{\Gamma}) = r + 1$.

We discuss two different cases below, depending on the cardinality of $U(\omega)$. We will use the following two events:

$$\mathcal{X} = \left\{ C \xleftrightarrow{R} \mathcal{N}(\overline{\Gamma}, r) \right\} \cap \left\{ C \xleftrightarrow{R} A \right\}^c$$

and

$$\mathcal{X}' = \left\{ C \xleftrightarrow{R} \mathcal{N}(\overline{\Gamma}, r) \right\} \cap \left\{ C \xleftrightarrow{R} A \right\}.$$

Our object is basically to show that $\mathbf{P}_p[\mathcal{X}] / \mathbf{P}_p[\mathcal{X}']$ is small, at least when $\mathbf{P}_p[\mathcal{X}] + \mathbf{P}_p[\mathcal{X}'] = \mathbf{P}_p\left[C \xleftrightarrow{R} \mathcal{N}(\overline{\Gamma}, r)\right]$ is not small.

Fact 1. There exists $C_1 < \infty$, depending only on ε , such that for any $t > 0$,

$$\mathbf{P}_p[\mathcal{X} \cap \{|U| \leq t\}] \leq (C_1)^t \mathbf{P}_p\left[\left(C \xleftrightarrow{R} \mathcal{N}(\overline{\Gamma}, r)\right)^c\right].$$

We prove this statement by constructing a disconnecting (or anti-gluing) map

$$\Phi : \mathcal{X} \cap \{|U| \leq t\} \rightarrow \left\{ C \xleftrightarrow{R} \mathcal{N}(\overline{\Gamma}, r) \right\}^c,$$

such that for any ω' in the image of Φ , the cardinality of its pre-image is bounded by a constant depending only on t .

We define $\Phi(\omega)$ by closing for every $z \in U(\omega)$, all the edges adjacent to a vertex in $\overline{B_r(z)}$ which are not in Γ . Observe that $\Phi(\omega)$ cannot contain any open path from C to $\mathcal{N}(\overline{\Gamma}, r)$. Let $|\overline{B_{r+1}}|$ denote the number of edges in $\overline{B_{r+1}}$. Lemma 3.5 can be applied with $s = 2t|\overline{B_{r+1}}|$ to yield

$$\mathbf{P}_p[\mathcal{X} \cap \{|U| \leq t\}] \leq \left(\frac{2}{p \wedge (1-p)}\right)^{2t|\overline{B_{r+1}}|} \mathbf{P}_p\left[\left(C \xleftrightarrow{R} \mathcal{N}(\overline{\Gamma}, r)\right)^c\right],$$

and we can conclude the proof of Fact 1 with $C_1 = (2/\varepsilon)^{2|\overline{B_{r+1}}|}$.

Fact 2. There exists $C_2 < \infty$, depending only on ε and k , such that for any $t > 8$,

$$\mathbf{P}_p[\mathcal{X} \cap \{|U| > t\}] \leq \frac{C_2}{t-8} \mathbf{P}_p[\mathcal{X}'].$$

We prove Fact 2 by constructing a map

$$\Phi : \mathcal{X} \cap \{|U| > t\} \rightarrow \mathfrak{P}(\mathcal{X}'),$$

such that for any $\omega' \in \mathcal{X}'$, the ω 's with $\omega' \in \Phi(\omega)$ agree on all but at most s specified edges, with s a constant depending only on k .

The construction of Φ is similar to that in [DST15a], as we now describe. For any $z \in U(\omega)$ that is not one of the eight corners of S , we will construct a new configuration

$\omega^{(z)}$ and define $\Phi(\omega) = \{\omega^{(z)} : z \in U(\omega), z \text{ is not a corner of } S\}$. The new configuration $\omega^{(z)}$ is constructed by the following three steps.

- (1) Define u', v' to be respectively the first and last vertices (when going from A to B) of $\Gamma(\omega)$ which are in $\overline{B_{r+1}(z)}$. Choose w' on the boundary of $\overline{B_{r+1}(z)}$, such that there exists an open self-avoiding path π (which could be a singleton) from w' to C . By the definition of $U(\omega)$ and \mathcal{X} , w' is distinct from u', z and v' . Choose u, v, w such that $(z, u), (z, v)$ and (z, w) are three distinct edges with $v \prec w$. If $z = u$ or z is a neighbor of u , we simply take $u = u'$, and if $z = v$, we take $v = v' = z$. Otherwise, u, v, w are chosen to be distinct sites from u', v', w' . (Note that this is possible because for z that is not a corner of S , the degree of z is at least 4. And since A, B, C are at least distance $r + 2$ apart, at most one of u', v', w' can be a neighbor of z).
- (2) Close all edges of ω in $\overline{B_{r+2}(z)}$ except the edges of $\overline{B_{r+2}(z)} \setminus \overline{B_{r+1}(z)}$ which are in $\Gamma(\omega)$ or π .
- (3) Open the edges $(z, u), (z, v), (z, w)$, together with three disjoint self-avoiding paths $\gamma_u, \gamma_v, \gamma_w$ inside $\overline{B_{r+1}(z)}$ connecting u to u', v to v' and w to w' .

By construction, $\omega^{(z)} \in \mathcal{X}'$. Now given ω' in the image of Φ , by the same argument as in [DST15a], z is the only site in the new minimal path $\Gamma(\omega^{(z)})$ (from A to B) that is connected to C without using any edge in $\Gamma(\omega^{(z)})$. Since $C \xrightarrow{R} A$, the path $\Gamma(\omega^{(z)})$ agrees with $\Gamma(\omega)$ up to u' . Then, because v is minimal among u, v, w, x , $\Gamma(\omega^{(z)})$ still goes through v' , and then agrees with $\Gamma(\omega)$ from v' to the end. Therefore $\omega' = \omega^{(z)}$ for some z that can be uniquely determined by ω' . Thus, the number of edges in $\{\omega : \text{such that } \omega' \in \Phi(\omega)\}$ that can vary is bounded by the number of edges in $\overline{B_{r+2}(z)}$. This shows that Φ satisfies the conditions of Lemma 3.5, with $s = |\overline{B_{r+2}}|$. This proves Fact 2 with $C_2 = (2/\varepsilon)^{|\overline{B_{r+2}}|}$.

To complete the first part of the proof, we set $x = \mathbf{P}_p[C \xrightarrow{R} \mathcal{N}(\bar{\Gamma}, r)]$, and combine Facts 1 and 2 to construct h_1 . Notice that

$$\mathbf{P}_p[\mathcal{X}] + \mathbf{P}_p[\mathcal{X}'] = x, \quad (3)$$

and Fact 1 implies $\mathbf{P}_p[\mathcal{X} \cap \{|U| \leq t\}] \leq (C_1)^t (1 - x)$. Together with Fact 2, this implies that

$$\mathbf{P}_p[\mathcal{X}'] \geq \frac{x - (C_1)^t (1 - x)}{1 + C_2/(t - 8)}.$$

Setting $t = \log |\log(1 - x)|$ in the equation above, one can easily construct $\delta > 0$ and a function $h_1 : [1 - \delta, 1] \rightarrow [0, 1]$ that is continuous, strictly increasing on $[1 - \delta, 1]$, with $h_1(1 - \delta) = 0$ and $h_1(1) = 1$. Set then $h_1(x) = 0$, for $x < 1 - \delta$. This ends the first part of the proof.

For the second part, we claim that for all $x \in [0, 1]$, one can take $h_2(x) = c_0 x$. Indeed, we can construct a map $\Phi : \mathcal{X} \rightarrow \mathfrak{P}(\mathcal{X}')$ by repeating the same construction as in the proof of Fact 2 but without needing to consider the cardinality of U . Since for any $\omega' \in \mathcal{X}'$, the number of edges in $\{\omega : \omega' \in \Phi(\omega)\}$ that can vary is bounded, this gives $\mathbf{P}_p[\mathcal{X}] \leq C_3 \mathbf{P}_p[\mathcal{X}']$. Together with (3), we conclude that we can take $h_2(x) = (1 + C_3)^{-1} x$. \square

Proof of Theorem 3.6. Set $\Gamma = \Gamma_{\min}^S(A, B)$. To see how the proof of Theorem 3.7 implies Theorem 3.6, we first note that the assumption in Theorem 3.7 that the projections of A, B, C on \mathbb{Z}^2 are at least sup-norm distance $r + 1$ apart is used to deal with the issue of the r -neighborhood of $\bar{\Gamma}$ in (2). Indeed, as long as their projections on \mathbb{Z}^2 are disjoint, by essentially the same proof as the one used for Theorem 3.7, we have

$$\mathbf{P}_p [C \xleftrightarrow{R} A] \geq \mathbf{h}_0(\mathbf{P}_p [C \xleftrightarrow{R} \bar{\Gamma}]). \quad (4)$$

We next note that when $R = S$, the event $\{A \xleftrightarrow{S} B, C \xleftrightarrow{S} D\}$ implies $\{C \xleftrightarrow{S} \bar{\Gamma}\}$, so by the Harris-FKG inequality,

$$\mathbf{P}_p [C \xleftrightarrow{S} \bar{\Gamma}] \geq (\mathbf{P}_p [A \xleftrightarrow{S} B] \wedge \mathbf{P}_p [C \xleftrightarrow{S} D])^2.$$

Combining the last inequality with (4) yields (1). \square

Finally we conclude with a last gluing lemma, that will allow us to glue together circuits. As we will see in the proof, it will be easier to glue a circuit with a path than gluing two paths. This is due to the fact that the local modification performed in this case does not create a new circuit, and the reconstruction step is easier.

We now define a total ordering on circuits. The specific choice of the ordering is not important, ours is the following. A circuit is basically a path $(\Gamma(i))_{i=1}^r$ in \mathbb{S}_k , such that $\Gamma(1) = \Gamma(r)$, and $(\Gamma(i))_{i=1}^{r-1}$ is a self avoiding path. Since we will identify circuits that differ by cyclic permutations or reverse orderings of their indices, we will assume that the representative self avoiding path $(\Gamma(i))_{i=1}^{r-1}$ has $\Gamma(1) \prec \Gamma(i)$ for $i > 1$ and has $\Gamma(2) \prec \Gamma(r-1)$. Given two circuits $\Gamma = (\Gamma(i))_{i=1}^{r_1}$, $\Gamma' = (\Gamma'(i))_{i=1}^{r_2}$ in $\overline{A_{a,b}}$ that surround the origin (i.e., their projections on \mathbb{Z}^2 have nonzero winding number around the origin), we set $\Gamma \prec \Gamma'$ by using the same lexicographical ordering as we defined before for self-avoiding paths.

The following statement will be used in the renormalization argument to prove Lemma 3.11.

Theorem 3.8. *Fix $\varepsilon > 0$ and $k \geq 1$. There exists $\mathbf{h}_1 \in \mathcal{H}$ such that for every $p \in [\varepsilon, 1 - \varepsilon]$ and $n \geq m \geq 3$,*

$$\mathbf{P}_p [\Gamma_1 \xleftrightarrow{R} \Gamma_2] \geq \mathbf{h}_1(f(3n, 2m)a(m, n)^2),$$

where $R = [0, 3n] \times [-m, m]$, Γ_1 is the minimal open circuit in $\overline{A_{m,n}}$ surrounding $\overline{B_m}$ ($\Gamma_1 = \emptyset$ if there is no such circuit), Γ_2 is the minimal open circuit in $\overline{A_{m,n}((3n, 0))}$ surrounding $\overline{B_m((3n, 0))}$, and $a(m, n)$ denotes the probability under \mathbf{P}_p that Γ_1 exists and is not empty.

Proof. We proceed in two steps. First, we prove

$$\mathbf{P}_p [\Gamma_1 \xleftrightarrow{R} \bar{\Gamma}_2] \geq \mathbf{h}(\mathbf{P}_p [\bar{\Gamma}_1 \xleftrightarrow{R} \bar{\Gamma}_2]), \quad (5)$$

and then

$$\mathbf{P}_p [\Gamma_1 \xleftrightarrow{R} \Gamma_2] \geq \mathbf{h}(\mathbf{P}_p [\Gamma_1 \xleftrightarrow{R} \bar{\Gamma}_2]). \quad (6)$$

We finish the proof by using the FKG inequality, which implies

$$\mathbf{P}_p \left[\overline{\Gamma_1} \xleftrightarrow{R} \overline{\Gamma_2} \right] \geq f(3n, 2m)a(m, n)^2.$$

Let us begin with the proof for (5). Given a configuration ω , we define $U(\omega)$ as the set of points $z \in R$ such that

- $z \in \overline{\Gamma_1(\omega)}$, and
- z is connected to $\overline{\Gamma_2(\omega)}$ by a self-avoiding path γ_z in R .

Let $x = f(3n, 2m)a(m, n)^2$, then by the same argument as used for Fact 1 in the proof of Theorem 3.7, we can show that there is some $C_1 < \infty$, such that

$$\mathbf{P}_p \left[\overline{\Gamma_1} \xleftrightarrow{R} \overline{\Gamma_2}, \Gamma_1 \xleftrightarrow{R} \overline{\Gamma_2}, |U| \leq t \right] \leq (C_1)^t (1 - x). \quad (7)$$

We then prove that there exists some $C_2 < \infty$ such that for every $t \geq 1$,

$$\mathbf{P}_p \left[\overline{\Gamma_1} \xleftrightarrow{R} \overline{\Gamma_2}, \Gamma_1 \xleftrightarrow{R} \overline{\Gamma_2}, |U| \geq t \right] \leq \frac{C_2}{t} \mathbf{P}_p \left[\overline{\Gamma_1} \xleftrightarrow{R} \overline{\Gamma_2} \right], \quad (8)$$

using a map

$$\Phi : \left| \begin{array}{ccc} \{\overline{\Gamma_1} \xleftrightarrow{R} \overline{\Gamma_2}, \Gamma_1 \xleftrightarrow{R} \overline{\Gamma_2}, |U| \geq t\} & \rightarrow & \mathfrak{P}(\{\overline{\Gamma_1} \xleftrightarrow{R} \overline{\Gamma_2}\}) \\ \omega & & \mapsto \{\omega^{(z)}, z \in U(\omega)\} \end{array} \right|.$$

Let $\omega \in \{\overline{\Gamma_1} \xleftrightarrow{R} \overline{\Gamma_2}, \Gamma_1 \xleftrightarrow{R} \overline{\Gamma_2}, |U| \geq t\}$. For every $z \in U(\omega)$, the configuration $\omega^{(z)} \in \{\overline{\Gamma_1} \xleftrightarrow{R} \overline{\Gamma_2}\}$ is constructed as follows.

- (1) Close all the edges in $\overline{B_1(z)}$ except those in $\Gamma_1(\omega)$ and γ_z .
- (2) Let $u \in \overline{z} \cap \Gamma_1(\omega)$ and $v \in \overline{z} \cap \gamma_z$, such that no vertex (except u and v) in the vertical segment between u and v belongs to $\Gamma_1(\omega)$ or γ_z . Then open all the vertical edges between u and v .

Denote by $\omega^{(z)}$ the resulting configuration. Observe that the local modification above does not create any new circuit in $\overline{A_{m,n}}$. Otherwise, the new circuit would contain all the vertical edges between u and v , which would imply some site on $\Gamma_1(\omega)$ is connected (through $\gamma_z(\omega)$) to $\overline{\Gamma_2(\omega)}$, which contradicts $\omega \in \{\overline{\Gamma_1} \xleftrightarrow{R} \overline{\Gamma_2}\}$. Therefore one can reconstruct ω by noting that $u \in \overline{z}$ is the only site on $\Gamma_1(\omega^{(z)}) = \Gamma_1(\omega)$ that connects to $\overline{\Gamma_2(\omega^{(z)})} = \overline{\Gamma_2(\omega)}$ without using any other edges in Γ_1 . Applying Lemma 3.5 leads to (8) with $C_2 = (2/\varepsilon)^{|\overline{B_1}|}$.

From (7) and (8) we can conclude (5) by using the same argument as in the proof of Theorem 3.7.

Similarly, we can prove (6) by defining $U(\omega)$ as the set of points $z \in R$ such that

- $z \in \overline{\Gamma_2(\omega)}$, and
- z is connected to $\Gamma_1(\omega)$ by a self-avoiding path γ_z in R .

And we construct a map

$$\Phi : \left| \begin{array}{ccc} \{\overline{\Gamma_1} \xleftrightarrow{R} \overline{\Gamma_2}, \Gamma_1 \xleftrightarrow{R} \Gamma_2, |U| \geq t\} & \rightarrow & \mathfrak{P}(\{\overline{\Gamma_1} \xleftrightarrow{R} \Gamma_2\}) \\ \omega & & \mapsto \{\omega^{(z)}, z \in U(\omega)\} \end{array} \right|.$$

Let $\omega \in \{\Gamma_1 \xleftrightarrow{R} \overline{\Gamma_2}, \Gamma_1 \not\xleftrightarrow{R} \Gamma_2, |U| \geq t\}$. For every $z \in U(\omega)$, the configuration $\omega^{(z)}$ is constructed as follows.

- (1) Close all the edges in $\overline{B_1(z)}$ except those in $\Gamma_2(\omega)$ and γ_z .
- (2) Let $u \in \overline{z} \cap \Gamma_2(\omega)$ and $v \in \overline{z} \cap \gamma_z$, such that no vertex (except u and v) in the vertical segment between u and v belongs to $\Gamma_2(\omega)$ or γ_z . Then open all the vertical edges between u and v .

As above, the local modification does not create any new circuit inside $\overline{A_{m,n}((3n, 0))}$, and one can reconstruct ω from $\omega^{(z)}$ by noting that $u \in \overline{z}$ is the only site on Γ_2 that connects to Γ_1 without using any other edges in Γ_2 . Applying Lemma 3.5, we obtain

$$\mathbf{P}_p \left[\Gamma_1 \xleftrightarrow{R} \overline{\Gamma_2}, \Gamma_1 \not\xleftrightarrow{R} \Gamma_2, |U| \geq t \right] \leq \frac{C_2}{t} \mathbf{P}_p \left[\Gamma_1 \xleftrightarrow{R} \Gamma_2 \right].$$

The same argument as in the proof of Theorem 3.7 yields (6). □

3.3. Crossing estimates. Let $R = \overline{[u, v] \times [w, t]}$ be a rectangular region in \mathbb{S}_k . Let $\mathbf{L}(R)$, $\mathbf{R}(R)$, $\mathbf{T}(R)$ and $\mathbf{B}(R)$ be respectively the left, right, top and bottom sides of R .

The following proposition extends to slabs some standard estimates in planar percolation.

Proposition 3.9. *Let r be as in Definition 3.1. Fix $\varepsilon > 0$ and $k \geq 1$. There exists $h_2 \in \mathcal{H}$ such that for every $p \in [\varepsilon, 1 - \varepsilon]$, for every $\kappa > 0$, $j \geq 2$, and every $n \geq r + 2$,*

1. $f_p(n + j\kappa n, n) \geq h_2^{j-1}(f_p(n + \kappa n, n))$,
2. $f_p(n, n + \kappa n) \geq h_2^{j-1}(f_p(n, n + j\kappa n))$,

where $h^j = \underbrace{h \circ \dots \circ h}_{j \text{ times}}$ denotes the j -th iterate of h .

Proof. Let us begin with Item 1. We only prove the $j = 2$ case,

$$f_p(n + 2\kappa n, n) \geq h_2(f_p(n + \kappa n, n)). \quad (9)$$

The more general statement (in fact a stronger result) follows by induction. For simplicity we assume that κn and $n/2$ are integers. Let $R = \overline{[0, n + \kappa n] \times [0, n]}$, $S = \overline{[0, n]^2}$ and $X = \overline{[0, n/2] \times \{0\}}$. Invariance under reflection and the square root trick imply

$$\begin{aligned} \mathbf{P}_p \left[X \xleftrightarrow{S} \mathbf{T}(S) \right] &\geq 1 - \sqrt{1 - f_p(n, n)} \\ &\geq 1 - \sqrt{1 - f_p(n + \kappa n, n)}. \end{aligned}$$

Then, by the gluing lemma of Theorem 3.6, we have

$$\begin{aligned} \mathbf{P}_p \left[X \xleftrightarrow{R} \mathbf{R}(R) \right] &\geq h_0(1 - \sqrt{1 - f_p(n + \kappa n, n)}) \\ &= h(f_p(n + \kappa n, n)), \end{aligned}$$

where we use that $1 - \sqrt{1 - f} \leq f$ for $f \in [0, 1]$, and define $h(f) \doteq h_0(1 - \sqrt{1 - f})$.

Next let $R' = \overline{[-\kappa n, n + \kappa n] \times [0, n]}$ and $Y = \overline{[n/2, n] \times \{0\}}$. Another application of the Theorem 3.6 gluing lemma inside R' gives

$$\begin{aligned} f_p(n + 2\kappa n, n) &= \mathbf{P}_p \left[\mathbf{L}(R') \xleftrightarrow{R'} \mathbf{R}(R) \right] \\ &\geq \mathbf{h}_0 \left(\mathbf{P}_p \left[\mathbf{L}(R') \xleftrightarrow{R'} Y \right] \wedge \mathbf{P}_p \left[X \xleftrightarrow{R'} \mathbf{R}(R') \right] \right) \\ &= \mathbf{h}_0 \left(\mathbf{h}(f_p(n + \kappa n, n)) \right), \end{aligned}$$

which gives exactly the statement of Eq. (9) with $\mathbf{h}_2 = \mathbf{h}_0 \circ \mathbf{h}$.

We now prove the second item. As we did for the first item we only prove

$$f_p(n, n + \kappa n) \geq \mathbf{h}_2(f_p(n, n + 2\kappa n)),$$

and the general statement follows by induction. Consider the event that there exists a top-down open crossing in R' . Then it is not hard to see that at least one of the following three events must occur:

- The rectangular region $\overline{[-\kappa n, n] \times [0, n]}$ is crossed from top to bottom;
- the rectangular region R is crossed from top to bottom;
- the square region S is crossed from left to right.

The maximum probability of these three events is at least $f_p(n, n + \kappa n)$, and the square root trick (Corollary 3.4) then gives

$$f_p(n, n + \kappa n) \geq 1 - (1 - \mathbf{P}_p[U])^{1/3} \geq 1 - (1 - f_p(n, n + 2\kappa n))^{1/3},$$

where U denotes the union of the three events. □

The next theorem allows us to create an open circuit in an annulus with positive probability. Before stating the theorem, we note that by elementary arguments (e.g., by bounding the expected number of open self-avoiding paths of length m starting from a given vertex), it is easy to see that there is some $\varepsilon > 0$ (depending only on k) such that $\sup_{n \geq 2} (f_\varepsilon(2n, n - 1)) < 1/2$.

Theorem 3.10. *Let $r \geq 3$ be as in Definition 3.1. Fix $k \geq 1$, and $\varepsilon > 0$ such that $\sup_{n \geq 2} (f_\varepsilon(2n, n - 1)) < 1/2$. For every $c > 0$, there exists $\lambda = \lambda(c) \geq 1$ and $c' > 0$ such that the following holds. For every $p \in [\varepsilon, 1 - \varepsilon]$ and every $n \geq 4r$,*

$$f_p(2n, n - 1) \geq c \implies \mathbf{P}_p[\mathcal{A}_{\lambda n, 2\lambda n}] \geq c'.$$

where $\mathcal{A}_{\ell, 2\ell}$ is the event that there exists inside $\overline{A_{\ell, 2\ell}}$ an open circuit surrounding $\overline{B_\ell}$.

Proof. Fix $p \in [\varepsilon, 1 - \varepsilon]$, $n \geq 4r$ and assume that

$$f_p(2n, n - 1) \geq c. \tag{10}$$

We may also add the restriction that

$$f_p(2n, n - 1) \leq 1/2. \tag{11}$$

Indeed, if (11) does not hold, one can lower the value of p in such a way that both (10) and (11) hold. The full conclusion then follows from the monotonicity of $\mathbf{P}_p[\mathcal{A}_{\lambda n, 2\lambda n}]$ in p .

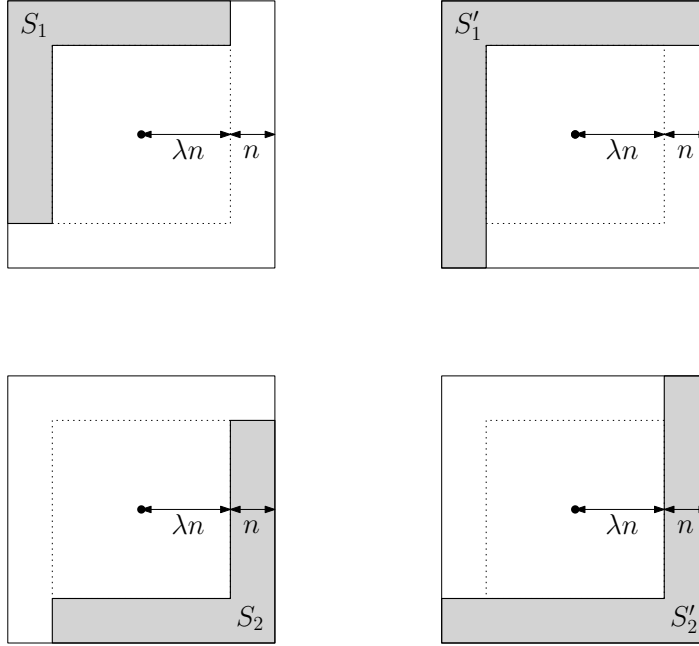


FIGURE 1. Illustration of the four L-shapes used in the proof.

Define the following L-shaped regions, illustrated in Fig. 1.

$$S_1 = \overline{[-\lambda n - n, \lambda n] \times (\lambda n, \lambda n + n]} \cup \overline{[-\lambda n - n, -\lambda n) \times [-\lambda n, \lambda n + n]},$$

$$S'_1 = \overline{[-\lambda n - n, \lambda n + n] \times (\lambda n, \lambda n + n]} \cup \overline{[-\lambda n - n, -\lambda n) \times [-\lambda n - n, \lambda n + n]}.$$

We also define

$$S_2 = \overline{[-\lambda n, \lambda n + n] \times [-\lambda n - n, -\lambda n)} \cup \overline{(\lambda n, \lambda n + n] \times [-\lambda n - n, \lambda n]},$$

$$S'_2 = \overline{[-\lambda n - n, \lambda n + n] \times [-\lambda n - n, -\lambda n)} \cup \overline{(\lambda n, \lambda n + n] \times [-\lambda n - n, \lambda n + n]}.$$

as the images of S_1 and S'_1 under the π -rotation around the origin.

Define then $\mathbf{B}(S_1) = \overline{[-\lambda n - n, -\lambda n) \times \{-\lambda n\}}$, $\mathbf{B}(S'_1) = \overline{[-\lambda n - n, -\lambda n) \times \{-\lambda n - n\}}$, $\mathbf{R}(S_1) = \overline{\{\lambda n\} \times (\lambda n, \lambda n + n]}$, $\mathbf{R}(S'_1) = \overline{\{\lambda n + n\} \times (\lambda n, \lambda n + n]}$, and similarly, $\mathbf{L}(S_2) = \overline{\{-\lambda n\} \times [-\lambda n - n, -\lambda n)}$, $\mathbf{L}(S'_2) = \overline{\{-\lambda n - n\} \times [-\lambda n - n, -\lambda n)}$, $\mathbf{T}(S_2) = \overline{(\lambda n, \lambda n + n] \times \{\lambda n\}}$, $\mathbf{T}(S'_2) = \overline{(\lambda n, \lambda n + n] \times \{\lambda n + n\}}$.

The general idea in the proof is to use that with positive probability there exists a unique cluster crossing the L-shape S_1 , and a unique cluster crossing S_2 . Then we connect these two clusters at two different places in order to create an open circuit inside the annulus $\overline{A_{\lambda n, \lambda n + n}}$. These two connections will be obtained by two local modifications in the top-right and bottom-left corners of the annulus $\overline{A_{\lambda n, \lambda n + n}}$. The uniqueness of the two clusters

crossing S_1 and S_2 will be important to avoid that the two local modifications interact. In other words, the uniqueness requirements will prevent the second local modification from cutting the connection created by the first local modification.

We first claim that for every $\lambda > 0$ there exists a constant $c_1 = c_1(c) > 0$, such that

$$\mathbf{P}_p \left[\mathbf{B}(S'_1) \xleftrightarrow{S'_1} \mathbf{R}(S'_1) \right] \geq c_1 \mathbf{P}_p \left[\mathbf{B}(S_1) \xleftrightarrow{S_1} \mathbf{R}(S_1) \right]. \quad (12)$$

This inequality can be obtained by performing several gluing procedures similar to those used in the proof of Proposition 3.9, and for these gluing procedures we use Remark 3 after Theorem 3.7.

Fix $\lambda > 0$ large enough such that

$$2^{-\lambda} \leq c_1^2/4. \quad (13)$$

(The value of λ depends on c through the constant c_1 but does not depend on n .)

Now, we need to make precise what we mean by a unique cluster crossing S_i . For $i = 1, 2$, let \mathcal{U}_i be the event that there exists a unique cluster in the configuration restricted to S_i that intersects both ends of S_i (i.e., the bottom and right ends of S_1 or respectively the left and top ends of S_2). We define

$$\begin{aligned} \mathcal{E}_0 &= \{ \mathbf{B}(S'_1) \xleftrightarrow{S'_1} \mathbf{R}(S'_1), \mathbf{L}(S'_2) \xleftrightarrow{S'_2} \mathbf{T}(S'_2) \}, \text{ and} \\ \mathcal{E} &= \mathcal{E}_0 \cap \mathcal{U}_1 \cap \mathcal{U}_2. \end{aligned}$$

We wish to show that the event \mathcal{E} occurs with probability larger than some positive constant. First, by the union bound we have

$$\begin{aligned} \mathbf{P}_p[\mathcal{E}] &\geq \mathbf{P}_p[\mathcal{E}_0] - \mathbf{P}_p[\mathcal{E}_0 \setminus \mathcal{U}_1] - \mathbf{P}_p[\mathcal{E}_0 \setminus \mathcal{U}_2] \\ &= \mathbf{P}_p[\mathcal{E}_0] - 2\mathbf{P}_p[\mathcal{E}_0 \setminus \mathcal{U}_1]. \end{aligned} \quad (14)$$

Then, Eq. (12) and the Harris-FKG inequality imply

$$\mathbf{P}_p[\mathcal{E}_0] \geq c_1^2 \mathbf{P}_p \left[\mathbf{B}(S_1) \xleftrightarrow{S_1} \mathbf{R}(S_1) \right]^2. \quad (15)$$

Also, observe that the occurrence of the event $\mathcal{E}_0 \setminus \mathcal{U}_1$ implies the existence of

- two disjoint open paths from $\mathbf{B}(S_1)$ to $\mathbf{R}(S_1)$ inside S_1 ,
- an open path from $\mathbf{L}(S_2)$ to $\mathbf{T}(S_2)$ inside S_2 .

Using first independence and then the BK inequality (see [Gri99] for the definition of disjoint occurrence and the BK inequality), we find

$$\begin{aligned} \mathbf{P}_p[\mathcal{E}_0 \setminus \mathcal{U}_1] &\leq \mathbf{P}_p[S_1 \text{ crossed by two disjoint open paths}] \mathbf{P}_p \left[\mathbf{L}(S_2) \xleftrightarrow{S_2} \mathbf{T}(S_2) \right] \\ &\leq \mathbf{P}_p \left[\mathbf{B}(S_1) \xleftrightarrow{S_1} \mathbf{R}(S_1) \right]^2 \mathbf{P}_p \left[\mathbf{L}(S_2) \xleftrightarrow{S_2} \mathbf{T}(S_2) \right] \\ &\leq \frac{c_1^2}{4} \mathbf{P}_p \left[\mathbf{B}(S_1) \xleftrightarrow{S_1} \mathbf{R}(S_1) \right]^2. \end{aligned} \quad (16)$$

For the last inequality, we use that an open path from $\mathbf{L}(S_2)$ to $\mathbf{T}(S_2)$ inside S_2 must cross λ (actually, 2λ) disjoint $2n$ by $n - 1$ rectangles in the long direction, and each of

these crossings occur with probability less than $1/2$ by Equation (11). Therefore, our choice of λ in Eq. (13) gives $\mathbf{P}_p [\mathbf{L}(S_2) \xleftrightarrow{S_2} \mathbf{T}(S_2)] \leq 2^{-\lambda} \leq c_1^2/4$.

Plugging (15) and (16) in (14), we obtain

$$\mathbf{P}_p [\mathcal{E}] \geq \frac{c_1^2}{2} \mathbf{P}_p [\mathbf{B}(S_1) \xleftrightarrow{S_1} \mathbf{R}(S_1)]^2.$$

Finally, as in the proof of Proposition 3.9, we can use the estimate of Eq. (10) to show that $\mathbf{P}_p [\mathbf{B}(S_1) \xleftrightarrow{S_1} \mathbf{R}(S_1)] \geq h(c)$ for some $h \in \mathcal{H}$ (that depends on λ). Therefore, there exists a constant $c_2 = c_2(c, \lambda) > 0$ such that

$$\mathbf{P}_p [\mathcal{E}] \geq c_2.$$

We claim that there exists a constant $c_3 > 0$ such that $\mathbf{P}_p [\mathcal{A}_{\lambda n, \lambda n + n}] \geq c_3^2 \mathbf{P}_p [\mathcal{E}]$, which will then finish the proof. To show this, we now perform a two step gluing procedure in the square regions $R_1 \doteq (\lambda n, \lambda n + n]^2$ and $R_2 \doteq [-\lambda n - n, -\lambda n]^2$ to create an open circuit. When \mathcal{E} occurs, we denote by Γ_1 the minimal open self-avoiding path from $\mathbf{B}(S_1)$ to $\mathbf{R}(S_1')$ inside

$$\overline{[-\lambda n - n, \lambda n + n] \times (\lambda n, \lambda n + n]} \cup \overline{[-\lambda n - n, -\lambda n] \times [-\lambda n, \lambda n + n]}.$$

We first prove

$$\mathbf{P}_p [\Gamma_1 \xleftrightarrow{S_2'} \mathbf{L}(S_2'), \mathcal{E}] \geq c_3 \mathbf{P}_p [\mathcal{E}], \quad (17)$$

and then

$$\mathbf{P}_p [\mathcal{A}_{\lambda n, \lambda n + n}] \geq c_3 \mathbf{P}_p [\Gamma_1 \xleftrightarrow{S_2'} \mathbf{L}(S_2'), \mathcal{E}]. \quad (18)$$

Let us begin with the proof for (17). Similarly to the proof of Theorem 3.7, we construct a map

$$\Phi : \left\{ \begin{array}{ccc} \{\Gamma_1 \xleftrightarrow{S_2'} \mathbf{L}(S_2'), \mathcal{E}\} & \rightarrow & \{\Gamma_1 \xleftrightarrow{S_2'} \mathbf{L}(S_2'), \mathcal{E}\} \\ \omega & \mapsto & \omega^{(z)} \end{array} \right.,$$

where the configuration $\omega^{(z)}$ is defined as follows. Given $\omega \in \{\Gamma_1 \xleftrightarrow{S_2'} \mathbf{L}(S_2'), \mathcal{E}\}$, we first choose the minimal point $z \in R_1$ such that

- $z \in \overline{\Gamma_1(\omega)}$, and
- z is connected to $\mathbf{L}(S_2')$ by an open self-avoiding path in S_2' .

Then we construct the configuration $\omega^{(z)}$ by the following three steps:

- (1) Define u', v' to be respectively the first and last vertices (when going from $\mathbf{B}(S_1)$ to $\mathbf{R}(S_1')$) of $\Gamma_1(\omega)$ which are in $\overline{B_r(z)} \cap R_1$. Choose w' on the boundary of $\overline{B_r(z)} \cap R_1$, such that there exists an open self-avoiding path π from w' to $\mathbf{L}(S_2')$ inside S_2' . The points u', v' and w' are all distinct, and by Definition 3.1, we can choose u, v, w such that $(z, u), (z, v)$ and (z, w) are three distinct edges with $v \prec w$.
- (2) Close all the edges of ω in $\overline{B_{r+1}(z)} \cap [\lambda n, \lambda n + n]^2$ except the edges of $\overline{B_{r+1}(z)} \setminus \overline{B_r(z)} \cap [\lambda n, \lambda n + n]^2$ which are in Γ_1 and π .
- (3) Open the edges $(z, u), (z, v), (z, w)$, together with three disjoint self-avoiding paths $\gamma_u, \gamma_v, \gamma_w$ inside $\overline{B_r(z)} \cap R_1$ connecting u to u', v to v' and w to w' .

By construction, $\omega^{(z)} \in \{\Gamma_1 \xleftrightarrow{S'_2} \mathbf{L}(S'_2), \mathcal{E}\}$. The uniqueness of the cluster crossing S_1 implies that the path $\Gamma_1(\omega^{(z)})$ agrees with $\Gamma_1(\omega)$ from $\mathbf{B}(S_1)$ to u' , and we can reconstruct the point z by noting that z is the only site in $\Gamma_1(\omega^{(z)})$ that is connected to $\mathbf{L}(S'_2)$ without using any edge in $\Gamma_1(\omega^{(z)})$. Therefore Φ satisfies the conditions of Lemma 3.5, with $s = |\overline{B_{r+1}}|$. Applying Lemma 3.5 then yields (17).

We next move to the proof of (18). We define Γ_2 as the minimal open crossing from $\mathbf{T}(S_2)$ to $\mathbf{L}(S'_2)$ in

$$\overline{[-\lambda n - n, \lambda n + n] \times [-\lambda n - n, -\lambda n]} \cup \overline{(\lambda n, \lambda n + n] \times [-\lambda n - n, \lambda n]}.$$

As before, we construct a map

$$\Phi : \left| \begin{array}{ccc} \{\mathcal{A}_{\lambda n, \lambda n + n}^c, \Gamma_1 \xleftrightarrow{S'_2} \mathbf{L}(S'_2), \mathcal{E}\} & \rightarrow & \mathcal{A}_{\lambda n, \lambda n + n} \\ \omega & \mapsto & \omega^{(z)} \end{array} \right|.$$

Let $\omega \in \{\mathcal{A}_{\lambda n, \lambda n + n}^c, \Gamma_1 \xleftrightarrow{S'_2} \mathbf{L}(S'_2), \mathcal{E}\}$. We choose a point $z \in \overline{\Gamma_2(\omega)}$ which is connected to $\mathbf{R}(S_1)$ inside S'_1 . We construct the configuration $\omega^{(z)}$ by essentially the same three steps as we did in proving (17), the only difference is now we do local modifications in $\overline{B_{r+1}}(z) \cap \overline{[-\lambda n - n, -\lambda n]^2}$, and u', v' are defined respectively to be the first and last vertices (when going from $\mathbf{T}(S_2)$ to $\mathbf{L}(S'_2)$) of $\Gamma_2(\omega)$ which are in $\overline{B_r(z)} \cap R_2$, and w' is connected by a self-avoiding path to $\mathbf{R}(S_1)$ inside S'_1 .

To see that $\omega^{(z)} \in \mathcal{A}_{\lambda n, \lambda n + n}$, we first note that the local modification above does not change the 'unique clusters' inside S_1 and S_2 (they are measurable with respect to the edge variables in S_1 and S_2). Then, since $\omega \in \mathcal{A}_{\lambda n, \lambda n + n}^c$, the path $\Gamma_2(\omega)$ cannot be connected to $\mathbf{R}(S_1)$. Otherwise, the fact that $\Gamma_1 \xleftrightarrow{S'_2} \mathbf{L}(S'_2)$ and the uniqueness of the cluster in S_1 would imply the existence of a circuit in $\overline{A_{\lambda n, \lambda n + n}}$. Therefore $\Gamma_2(\omega^{(z)})$ agrees with $\Gamma_2(\omega)$ up to u' , and it must go through z and exit $\overline{B_r(z)}$ through v' . Then it agrees with $\Gamma_2(\omega)$ from v' to the end. Thus $\omega^{(z)} \in \mathcal{A}_{\lambda n, \lambda n + n}$, and one can reconstruct the point z by noting that z is the only site in $\Gamma_2(\omega^{(z)})$ that is connected to $\mathbf{R}(S_1)$ without using any edge in $\Gamma_2(\omega^{(z)})$. Therefore Φ satisfies the conditions of Lemma 3.5, with $s = |\overline{B_{r+1}}|$. Applying Lemma 3.5 then yields (18), and thus concludes the proof. \square

3.4. Renormalization inputs.

Lemma 3.11 (Finite criterion for $\theta(p) > 0$). *Fix $k \geq 0$ and $\varepsilon > 0$ such that $\varepsilon < p_c(\mathbb{S}_k) < 1 - \varepsilon$. There exists a constant $c_1 > 0$, such that the following holds. For every $p \in [\varepsilon, 1 - \varepsilon]$ and every $n \geq 4r$,*

$$f_p(2n, n - 1) > 1 - c_1 \implies \mathbf{P}_p \left[0 \xleftrightarrow{\mathbb{S}_k} \infty \right] > 0.$$

Proof. Fix $k \geq 0$ and $\varepsilon > 0$ as in the statement of the lemma. We first prove the following claim, which isolates the renormalization argument we are using.

Claim. *There exist $\eta > 0$ such that for every $p \in [\varepsilon, 1 - \varepsilon]$ and every $n \geq m \geq 4r$,*

$$\left. \begin{aligned} f_p(3n, 2m) &\geq 1 - \eta \\ \mathbf{P}_p[\mathcal{A}_{m,n}] &\geq 1 - \eta \end{aligned} \right\} \implies \mathbf{P}_p[0 \xleftrightarrow{\mathbb{S}_k} \infty] > 0. \quad (19)$$

Proof of Claim. Fix $p_0 < 1$ to be such that any 1-dependent bond percolation measure on \mathbb{Z}^2 with marginals larger than p_0 produces an infinite cluster. (This is well defined by standard stochastic domination arguments [LSS97] or a Peierls argument [BBW05]). Let $G = (V, E)$ be the graph with vertex set $V := 3n\mathbb{Z}^2$, and edge set $E := \{\{v, w\} : |v - w| = 3n\}$. It is a rescaled version of the standard two-dimensional grid \mathbb{Z}^2 .

We define a percolation process X on G follows. Consider a Bernoulli percolation process with density p on the slab \mathbb{S}_k . Let $e = \{u, v\} \in E$ be a horizontal edge with $v = u + (3n, 0)$. Set $X(e) = 1$ if

- There exists an open circuit in $\overline{A_{m,n}(u)}$ surrounding $\overline{B_m(u)}$, and an open circuit in $\overline{A_{m,n}(v)}$ surrounding $\overline{B_m(v)}$.
- the minimal open self-avoiding circuit inside $\overline{A_{m,n}(u)}$ is connected to the minimal open self-avoiding circuit inside $\overline{A_{m,n}(v)}$ by an open path that lies inside $\overline{u + ([0, 3n] \times [-m, m])}$.

Set $X(e) = 0$ otherwise. Define $X(e)$ analogously when e is a vertical edge.

By Theorem 3.8, the condition on the left hand side of (19) implies

$$\mathbf{P}_p[X(e) = 1] \geq h_1(f_p(3n, 2m)\mathbf{P}_p[\mathcal{A}_{m,n}]^2) \geq h_1((1 - \eta)^3).$$

If we choose η small enough, the above probability is larger than p_0 . Since the percolation process X is 1-dependent, there exists with positive probability an infinite self avoiding path in G made of edges e satisfying $X(e) = 1$. This implies that in the slab, we have $\mathbf{P}_p[0 \xleftrightarrow{\mathbb{S}_k} \infty] > 0$. This ends the proof of the claim. \square

We now prove the lemma. By Theorem 3.10 one can first choose two constants $\lambda > 0$ and $c' > 0$ such that for every $p \in [\varepsilon, 1 - \varepsilon]$ and every $n \geq 4r$

$$f_p(2n, n - 1) \geq 1/4 \implies \mathbf{P}_p[\mathcal{A}_{\lambda n, 2\lambda n}] \geq c'. \quad (20)$$

Then we choose a constant $\ell < \infty$ such that $(1 - c')^\ell < \eta$. By Item 1 of Proposition 3.9, we can finally choose $c_1 > 0$ small enough such that

$$f_p(2m, m - 1) > 1 - c_1 \implies f_p(2^\ell \lambda m, m - 1) \geq 1 - \eta \geq 1/4. \quad (21)$$

One can easily check that this choice of $c_1 > 0$ concludes the proof. Assume that for some $m \geq 4r$, $f_p(2m, m - 1) > 1 - c_1$. Eq. (21) and (20) give for every $0 \leq i \leq \ell - 1$, $\mathbf{P}_p[\mathcal{A}_{2^i \lambda m, 2^{i+1} \lambda m}] \geq c'$. Therefore, by independence we have

$$\mathbf{P}_p[\mathcal{A}_{m, 2^\ell \lambda m}] \geq 1 - (1 - c')^\ell \geq 1 - \eta.$$

The claim above applied to $n = 2^\ell \lambda m$ implies that $\mathbf{P}_p[0 \xleftrightarrow{\mathbb{S}_k} \infty] > 0$. \square

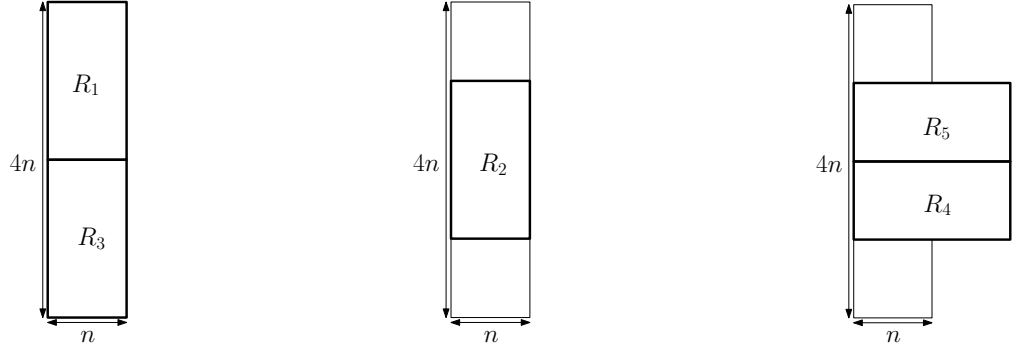


FIGURE 2. A covering of $[0, n] \times [0, 4n]$ by five n times $2n$ rectangles.

Lemma 3.12 (Finite criterion for exponential decay). *There exists an absolute constant $c_2 > 0$, such that $f_p(n, 2n) < c_2$ for some $n \geq 1$, implies that for every m , $\mathbf{P}_p \left[0 \xleftrightarrow{\mathbb{S}_k} \partial \overline{B_m} \right] \leq e^{-cm}$.*

Proof. This result is standard and can be proved in various ways. We present here a renormalization argument of Kesten [Kes82]. Let $n \geq 1$. If the rectangle $[0, 2n] \times [0, 4n]$ is crossed horizontally, then both rectangles $[0, n] \times [0, 4n]$ and $[n, 2n] \times [0, 4n]$ must be crossed horizontally. Using translation invariance and independence, we obtain

$$f_p(2n, 4n) \leq f_p(n, 4n)^2. \quad (22)$$

Now consider the covering of $[0, n] \times [0, 4n]$ by the five n times $2n$ rectangles R_1, \dots, R_5 illustrated on Fig. 2. If $[0, n] \times [0, 4n]$ is crossed horizontally then at least one of the five rectangles R_1, \dots, R_5 must be crossed in the easy direction. Using translation invariance and the union bound, we find

$$f_p(n, 4n) \leq 5f_p(n, 2n).$$

Together with Eq. (22) we obtain for every $n \geq 1$,

$$f_p(2n, 4n) \leq 25f_p(n, 2n)^2. \quad (23)$$

If $f_p(n_0, 2n_0) < 1$ for some $n_0 \geq 1$, Eq. (23) implies by induction that the sequence $f_p(n, 2n)$ decays exponentially fast in n , which easily implies that the probability for 0 to be connected to $\partial \overline{B_n}$ decays exponentially. \square

Lemma 3.13. *For critical Bernoulli percolation on \mathbb{S}_k (i.e., $p = p_c(\mathbb{S}_k)$) we have $f_p(2n, n-1) \leq 1 - c_1$ and $f_p(n, 2n) \geq c_2$ for every $n \geq 4r$.*

Proof. Take $\varepsilon > 0$ such that $p_c \in (\varepsilon, 1 - \varepsilon)$. Consider the set $\{p \in (\varepsilon, 1 - \varepsilon) : \text{there exists } n \geq 4r \text{ s.t. } f_p(2n, n-1) > 1 - c_1\}$. It is open and does not intersect $[0, p_c(\mathbb{S}_k))$ (by Lemma 3.11). Thus, p_c does not belong to this set, and therefore $f_p(2n, n-1) \leq 1 - c_1$ for every $n \geq 4r$ and $p \leq p_c$. Similarly, the inequality $f_p(n, 2n) \geq c_2$ at $p = p_c$ follows from Lemma 3.12. \square

3.5. The RSW-Theorem: positive probability version.

Theorem 3.14. *Fix $\varepsilon > 0$ and $k \geq 1$. For $p \in [\varepsilon, 1 - \varepsilon]$, the following implication holds for the horizontal crossing probability $f(m, n) = f_p(m, n)$.*

$$\text{If } \inf_{n \geq 1} f(n, 2n) > 0, \text{ then } \inf_{n \geq 1} f(2n, n) > 0.$$

All this section is devoted to the proof of this theorem. Under the assumption of the theorem, we can choose a constant $c_0 > 0$ such that for every $n \geq 1$,

$$f(n, 2n) \geq c_0. \quad (24)$$

We will use several gluing lemmas presented in Section 3.2. To this end, we fix a number $r \geq 3$ as in Definition 3.1. In the proof below we use various constants denoted c_j , each is independent of n .

Define $S = [0, 7n] \times [0, 8n]$, $R = [-7n, 7n] \times [0, 13n]$, $X = \{7n\} \times [0, 4n]$ and $Y = \{7n\} \times [5n, 13n]$ (see Fig. 3 for an illustration). Let \mathcal{A} be the event that there exist

- an open path in S from its left side to X and
- an open path in R from its bottom side to Y .

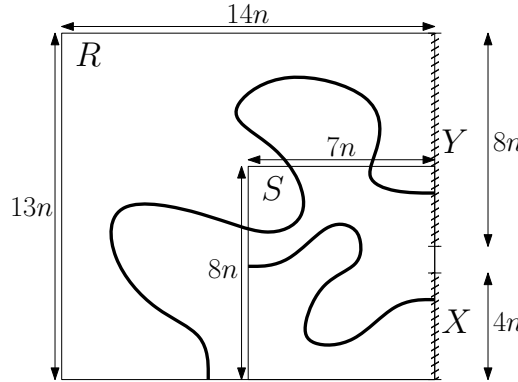


FIGURE 3. Diagrammatic representation of the event \mathcal{A} .

Lemma 3.15. *Assume that Eq. (24) holds. Then there exists a constant $c_4 > 0$ independent of $n \geq 4r$ such that*

$$\mathbf{P}_p[\mathcal{A}] \geq c_4.$$

Proof. We will prove that there exist constants $c_1, c_3 > 0$ such that

$$\mathbf{P}_p[\mathbf{L}(S) \xrightarrow{S} X] \geq c_1, \text{ and} \quad (25)$$

$$\mathbf{P}_p[\mathbf{B}(R) \xrightarrow{R} Y] \geq c_3. \quad (26)$$

Lemma 3.15 then follows by the Harris-FKG inequality with $c_4 = c_1 c_3$.

By Eq. (24), we have $f(7n, 14n) \geq c_0$. Therefore, by Item 2 of Proposition 3.9 (with $\kappa = 1/7$ and $j = 7$), we have $f(7n, 8n) \geq h_2^6(c_0) > 0$. In other words, the rectangle

S is crossed horizontally by an open path with probability larger than $h_2^6(c_0)$. Using a symmetry and the union bound, we obtain Eq. (25) with $c_1 = h_2^6(c_0)/2$.

Let us now prove Eq. (26). Since $f(14n, 28n)$ and $f(13n, 26n)$ are at least c_0 , Item 2 of Proposition 3.9 implies

$$f(14n, 16n) \geq h_2^6(c_0) \geq c_2, \text{ and} \quad (27)$$

$$f(13n, 14n) \geq h_2^{12}(c_0) \geq c_2, \quad (28)$$

where $c_2 = \min\{h_2^6(c_0), h_2^{12}(c_0)\}$. Consider the event that there exists inside $K = [-7n, 7n] \times [-3n, 13n]$ an open path Π from $L(K)$ to Y . Note that Y is the top half of $R(K)$. By (27) and a symmetry, this occurs with probability larger than $c_2/2$. When the path Π exists, either it touches the bottom side of R , or it remains inside R . Hence, by the union bound, at least one of the following two cases holds:

- $\mathbf{P}_p \left[\mathbf{B}(R) \xleftrightarrow{R} Y \right] \geq c_2/4$;
- $\mathbf{P}_p \left[\mathbf{L}(R) \xleftrightarrow{R} Y \right] \geq c_2/4$.

The first case gives exactly (26). In the second case we can conclude the proof by using the Theorem 3.6 gluing lemma inside R . We would know that R is crossed from left to right by an open path with probability larger than $c_2/4$, and from top to bottom with probability larger than c_2 (by Eq. (28)). Theorem 3.6 would then imply Eq. (26) with $c_3 = h_0(c_2/4)$. \square

We now investigate a possible geometry of connecting paths when \mathcal{A} occurs, which will be used near the end of the proof of Theorem 3.14. Let γ be a deterministic path from X to $L(S)$ in S such that $\gamma \cap Y = \emptyset$. Write γ' for the symmetric reflection of γ through the plane $\{0\} \times \mathbb{R}^2$. Notice that the set $\overline{\gamma \cup \gamma'}$ disconnects the top side $\mathbf{T}(R)$ of R from its bottom side $\mathbf{B}(R)$, in the sense that any path from top to bottom in R must intersect $\overline{\gamma \cup \gamma'}$. Let $K_0(\gamma)$ be the connected component of $\mathbf{T}(R)$ in $R \setminus \overline{\gamma \cup \gamma'}$. Then, set $K(\gamma) = (K_0(\gamma) \cup \partial K_0(\gamma)) \setminus \mathcal{N}(\overline{\gamma}, 3r)$, where we note that every edge in the boundary, $\partial K_0(\gamma)$, of $K_0(\gamma)$ is between a vertex in $K_0(\gamma)$ and one in $\overline{\gamma \cup \gamma'}$. We recall that r is given by Definition 3.1, and $\mathcal{N}(\overline{\gamma}, 3r)$ is the set of vertices within sup-norm distance $3r$ of $\overline{\gamma}$. Define \mathcal{C}_γ as the event that there exists an open path in $K(\gamma)$ from Y to $\overline{\gamma'}$. (See Fig. 4.)

Lemma 3.16. *There exists $c_7 > 0$ such that for every $n \geq 6r$,*

$$\max_{\gamma} (\mathbf{P}[\mathcal{C}_\gamma]) \geq c_4/3 \implies f(14n, 13n) \geq c_7.$$

where the maximum is taken over all deterministic paths γ from X to $L(S)$ in S , such that $\gamma \cap Y = \emptyset$.

Proof. Take γ such that $\mathbf{P}[\mathcal{C}_\gamma] \geq c_4/3$. Let Y' denote the symmetric reflection of Y through the plane $\{0\} \times \mathbb{R}^2$. Assume for simplicity that n is a multiple of $2r$. Define

$$K_{\square} \doteq \bigcup_{z \in 2r\mathbb{Z}^2 \text{ s.t. } \overline{B_r(z)} \subset K(\gamma)} \overline{B_r(z)}.$$

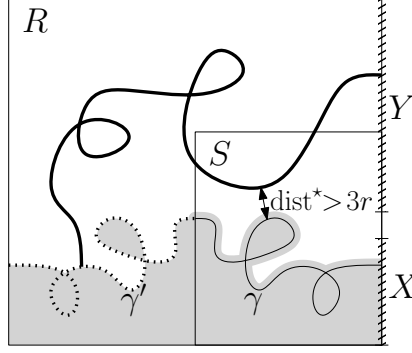


FIGURE 4. A diagrammatic representation of the event \mathcal{C}_γ that there is an open path from Y to $\bar{\gamma}'$ inside the region $K(\gamma)$ (which corresponds to the complement in R of the grey region).

We consider the left- and right-bottom parts of the boundary of K_\square , defined respectively by $A = \partial K_\square \cap \overline{(-7n, 0) \times [0, 13n]}$ and $A' = \partial K_\square \cap \overline{(0, 7n) \times [0, 13n]}$. Observe that

$$\mathbf{P}_p \left[A \xleftrightarrow{K_\square} Y \right] \geq \mathbf{P}[\mathcal{C}_\gamma] \geq c_4/3.$$

The domain K_\square is regular enough to apply Theorem 3.6 gluing Lemma (see Remark 2 after Theorem 3.7). We obtain

$$\mathbf{P}_p \left[Y \xleftrightarrow{K_\square} Y' \right] \geq \mathbf{h}_0(\mathbf{P}_p \left[A \xleftrightarrow{K_\square} Y \right] \wedge \mathbf{P}_p \left[A' \xleftrightarrow{K_\square} Y' \right]),$$

for some $\mathbf{h}_0 \in \mathcal{H}$. This gives

$$f(14n, 13n) \geq c_7,$$

with $c_7 = \mathbf{h}_0(c_4/3)$, which concludes the proof. \square

Proof of Theorem 3.14. Assume that Eq. (24) holds. By Lemma 3.15 we have $\mathbf{P}_p[\mathcal{A}] \geq c_4$.

Let $\Gamma = \Gamma_{\min}^S(X, \mathbf{L}(S))$ be the minimal open path from X to $\mathbf{L}(S)$ inside S . (Recall that \mathcal{A} does not occur if there is no such path.) Let \mathcal{B}_1 be the event that there exists an open path from Y to X inside R and \mathcal{B}_2 be the event that there exists an open path from Y to $\mathcal{N}(\bar{\Gamma}, 3r)$ inside R (see Fig. 5).

By the union bound, we have

$$c_4 \leq \mathbf{P}_p[\mathcal{B}_1] + \mathbf{P}_p[\mathcal{B}_2] + \mathbf{P}_p[\mathcal{A} \cap \mathcal{B}_1^c \cap \mathcal{B}_2^c].$$

At least one of the three terms on the right hand side must be larger than $c_4/3$, and we distinguish between these three cases. The argument for the third case will use Lemma 3.16.

Case 1: $\mathbf{P}_p[\mathcal{B}_1] \geq c_4/3$.

Let $Z_i = \{7n\} \times [in, (i+1)n)$ for $i \in \mathbb{N}$. Since $X = Z_0 \cup \dots \cup Z_3$ and $Y = Z_5 \cup \dots \cup Z_{12}$, we have

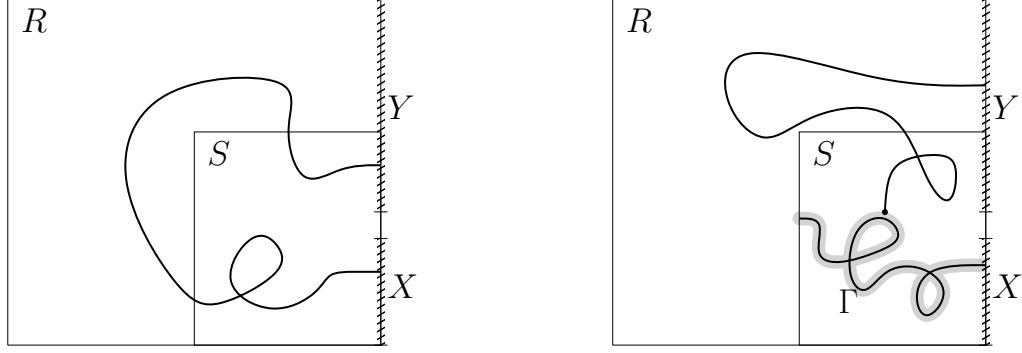


FIGURE 5. Diagrammatic representations of the event \mathcal{B}_1 (on the left) and the event \mathcal{B}_2 (on the right).

$$c_4/3 \leq \mathbf{P}_p[\mathcal{B}_1] \leq \sum_{\substack{0 \leq i \leq 3 \\ 5 \leq j \leq 12}} \mathbf{P}_p[Z_i \xleftrightarrow{R} Z_j].$$

Therefore there exists some i, j with $0 \leq i \leq 3$ and $5 \leq j \leq 12$ such that $\mathbf{P}_p[Z_i \xleftrightarrow{R} Z_j] \geq c_4/96$. We assume that

$$\mathbf{P}_p[Z_3 \xleftrightarrow{R} Z_5] \geq c_4/96,$$

since the other cases can be treated similarly. Let $R_1 = [-7n, 7n] \times [0, 40n]$. By the Theorem 3.6 gluing lemma and translation invariance, we have for every $6 \leq i \leq 32$,

$$\begin{aligned} \mathbf{P}_p[Z_3 \xleftrightarrow{R_1} Z_i] &\geq \mathbf{h}_0(\mathbf{P}_p[Z_3 \xleftrightarrow{R_1} Z_{i-1}] \wedge \mathbf{P}_p[Z_{i-2} \xleftrightarrow{R_1} Z_i]) \\ &\geq \mathbf{h}_0(\mathbf{P}_p[Z_3 \xleftrightarrow{R_1} Z_{i-1}] \wedge \mathbf{P}_p[Z_3 \xleftrightarrow{R} Z_5]) \\ &\geq \mathbf{h}_0(\mathbf{P}_p[Z_3 \xleftrightarrow{R_1} Z_{i-1}] \wedge (c_4/96)). \end{aligned}$$

By induction, this implies that Z_3 is connected to Z_{32} inside R_1 with probability larger than $c_5 := \mathbf{h}_0^{27}(c_4/96)$. When this holds, the rectangle $[-7n, 7n] \times [4n, 32n]$ is crossed from top to bottom by an open path. Therefore,

$$f(28n, 14n) \geq c_5.$$

Case 2: $\mathbf{P}_p[\mathcal{B}_2] \geq c_4/3$.

Apply the Theorem 3.7 gluing lemma, we have

$$\mathbf{P}_p[\mathcal{B}_1] \geq \mathbf{h}_0(\mathbf{P}_p[Y \xleftrightarrow{R} \mathcal{N}(\bar{\Gamma}, 3r)]) = \mathbf{h}_0(\mathbf{P}_p[\mathcal{B}_2]) \geq \mathbf{h}_0(c_4/3).$$

Then, as in the first case, there exists a constant $c_6 > 0$ such that

$$f(28n, 14n) \geq c_6.$$

Case 3: $\mathbf{P}_p[\mathcal{A} \cap \mathcal{B}_1^c \cap \mathcal{B}_2^c] \geq c_4/3$.

When $\mathcal{A} \cap \mathcal{B}_1^c \cap \mathcal{B}_2^c$ holds the open path from $\mathbf{B}(R)$ to Y must be at distance at least $3r + 1$ from the minimal path $\Gamma = \Gamma_{\min}^S(X, \mathbf{L}(S))$. Define \mathcal{C} to be the set of vertices that are either connected to X inside R or connected to a point z whose distance from $\bar{\Gamma}$ satisfies $\text{dist}^*(z, \bar{\Gamma}) \leq 3r$. Alternatively, the set \mathcal{C} can be defined by the following two-step exploration. First, explore all the open clusters touching X and notice that the minimal path Γ is already determined after this first exploration. In a second step, explore the open clusters of all the vertices in the $3r$ -neighborhood of $\bar{\Gamma}$ (that have not been explored yet). We make two observations:

- (a) If the event $\mathcal{A} \cap \mathcal{B}_1^c \cap \mathcal{B}_2^c$ occurs, then there exists an open path from Y to $\mathbf{B}(R)$ and the random set \mathcal{C} does not intersect Y (otherwise \mathcal{B}_1 or \mathcal{B}_2 occurs). Therefore, the open path from Y to $\mathbf{B}(R)$ must lie in $R \setminus \mathcal{C}$. Therefore, the event $Y \xrightarrow{R \setminus \mathcal{C}} \mathbf{B}(R)$ occurs.
- (b) If C is an admissible value for \mathcal{C} , the event $\mathcal{C} = C$ is measurable with respect to the status of the edges adjacent to the set C . In particular, the status of the edges in $R \setminus C$ is independent of the event $\mathcal{C} = C$.

Summing over the admissible realizations for the pair (\mathcal{C}, Γ) which allow the event $\mathcal{A} \cap \mathcal{B}_1^c \cap \mathcal{B}_2^c$ to occur, we obtain

$$\begin{aligned}
 c_4/3 \leq \mathbf{P}_p[\mathcal{A} \cap \mathcal{B}_1^c \cap \mathcal{B}_2^c] &= \sum_{(C, \gamma)} \mathbf{P}_p[\mathcal{A} \cap \mathcal{B}_1^c \cap \mathcal{B}_2^c, \mathcal{C} = C, \Gamma = \gamma] \\
 &\stackrel{(a)}{\leq} \sum_{(C, \gamma)} \mathbf{P}_p\left[Y \xrightarrow{R \setminus C} \mathbf{B}(R), \mathcal{C} = C, \Gamma = \gamma\right] \\
 &\stackrel{(b)}{=} \sum_{(C, \gamma)} \mathbf{P}_p\left[Y \xrightarrow{R \setminus C} \mathbf{B}(R)\right] \mathbf{P}_p[\mathcal{C} = C, \Gamma = \gamma] \\
 &\leq \sum_{(C, \gamma)} \mathbf{P}_p[\mathcal{C}_\gamma] \mathbf{P}_p[\mathcal{C} = C, \Gamma = \gamma] \\
 &\leq \max_{\gamma} \mathbf{P}_p[\mathcal{C}_\gamma]. \tag{29}
 \end{aligned}$$

The definition of \mathcal{C}_γ and the next to last inequality here are explained in the discussion before Lemma 3.16 (see Fig. 4.) Equation (29) together with Lemma 3.16 imply

$$f(14n, 13n) \geq c_7.$$

Then, by Item 1 of Proposition 3.9 we obtain that $f(28n, 13n) \geq c_8 := h_2^{14}(c_7)$, which implies that

$$f(28n, 14n) \geq c_8.$$

Combining the three cases above, we have

$$\inf_{n \geq 1} f(28n, 14n) \geq c_9,$$

with $c_9 = \min\{c_5, c_6, c_8\}$, which (by another use of Proposition 3.9) yields the conclusion of Theorem 3.14. \square

3.6. RSW-Theorem: high-probability version.

Theorem 3.17. *Fix $\varepsilon > 0$ and $k \geq 1$. For $p \in [\varepsilon, 1 - \varepsilon]$, if $\sup_{n \geq 1} f(n, 2n) = 1$, then $\sup_{n \geq 1} f(2n, n) = 1$.*

Proof. Assume that $\sup_{n \geq 1} f(n, 2n) = 1$; then by Proposition 3.9,

$$\sup_{n \geq 1} f(3n, 4n) = 1. \quad (30)$$

By Lemma 3.12, we also have that

$$\inf_{n \geq 1} f(n, 2n) > 0; \quad (31)$$

otherwise, Lemma 3.12 would imply exponential decay of the one-arm event, which would contradict (30). By Theorem 3.14, Eq. (31) implies

$$\inf_{n \geq 1} f(2n, n) > 0.$$

Using Theorem 3.10 (and Item 1 of Proposition 3.9), we can fix a constant $c_0 > 0$ such that, for every $n \geq 1$,

$$\mathbf{P}_p[\text{there exists an open circuit in } \overline{A_{n,2n}} \text{ surrounding } \overline{B_n}] \geq c_0. \quad (32)$$

Fix $\delta > 0$. By Equation (32) and independence, there exists a constant $c_1 < \infty$ large enough such that for every $n \geq 1$ and every $z \in \mathbb{Z}^2$,

$$\mathbf{P}_p[\text{there exists an open circuit in } \overline{A_{n,c_1n}(z)} \text{ surrounding } \overline{B_n(z)}] > 1 - \delta, \quad (33)$$

Let $R = [1, 1 + 3c_1n] \times [-2c_1n, 2c_1n]$. By symmetry and the square root trick, there exists $y \in \{0, n, \dots, (2c_1 - 1)n\}$ such that $\overline{\{1\} \times [y, y + n]}$ is connected in R to the right side $\mathbf{R}(R)$ with probability larger than

$$1 - (1 - f(3c_1n, 4c_1n))^{1/4c_1}.$$

Therefore, by Equation (30), we can find an n such that

$$\mathbf{P}_p[\overline{\{1\} \times [y, y + n]} \xleftrightarrow{R} \mathbf{R}(R)] \geq 1 - \delta. \quad (34)$$

Consider the set $S = \overline{A_{n,c_1n}(z) \setminus (\{0\} \times [y, \infty))}$ with $z = (0, y + n/2)$. Define also the sets $A = \overline{\{1\} \times [y + n/2 + n, y + n/2 + c_1n]}$ and $B = \overline{\{-1\} \times [y + n/2 + n, y + n/2 + c_1n]}$. The key feature of these subsets (see Fig. 6) is that the existence of an open circuit in $\overline{A_{n,c_1n}(z)}$ with $z = (0, y + n/2)$ surrounding $\overline{B_n(z)}$ implies that there is an open path from A to B inside S .

Set $\Gamma = \Gamma_{\min}^S(A, B)$. Using Equations (33) and (34), and an adaptation of the Theorem 3.6 gluing lemma (where S is replaced by $R^* = [-3c_1n, 3c_1n] \times [-2c_1n, 3c_1n] \setminus (\{0\} \times [y, \infty))$ here), one has

$$\mathbf{P}_p[B \xleftrightarrow{R^*} \mathbf{R}(R)] \geq h_0(1 - \delta).$$

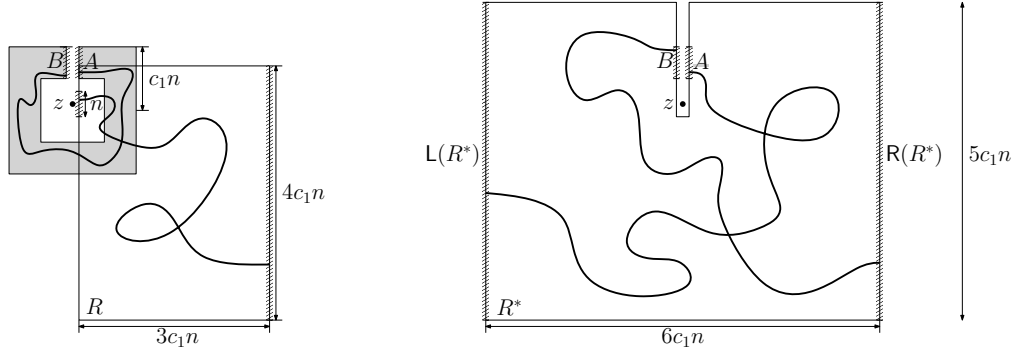


FIGURE 6. Illustration of the geometric construction used to create an open crossing inside R^* . The grey region corresponds to the set S . First, we use the two open paths illustrated on the left picture to create an open path from B to the right side of R . Then we use the two open paths illustrated on the right picture to create an open path from left to right in R^* .

Let R' be the symmetric reflection of R through the plane $\{0\} \times \mathbb{R}^2$. We have, using the Theorem 3.6 gluing lemma again, that

$$\begin{aligned} f(6c_1n, 5c_1n) &\geq \mathbf{h}_0(\mathbf{P}_p[B \xleftrightarrow{R^*} R(R)] \wedge \mathbf{P}_p[A \xleftrightarrow{R^*} L(R)]) \\ &\geq \mathbf{h}_0^2(1 - \delta). \end{aligned}$$

Since $\delta > 0$ was arbitrary, this completes the proof. \square

3.7. Proof of Theorem 3.1. By Lemma 3.13 and Theorem 3.14, we have

$$\inf_{n \geq 1} f(2n, n) > 0. \quad (35)$$

By Lemma 3.13, Item 2 of Proposition 3.9 and Theorem 3.17, we have

$$\sup_{n \geq 1} f(n, 2n) < 1. \quad (36)$$

Eqs. (35) and (36) together with Items 1 and 2 of Proposition 3.9 conclude the proof.

3.8. Proof of Corollary 3.2. The proofs of these items are standard. The first item follows from the RSW theorem and Theorem 3.10. For the second item, note that crossing the aspect ratio 2 annulus requires crossing an aspect ratio 4 rectangle, and then using the square root trick and Theorem 3.1 completes the proof. For the third item, write $\overline{B}_n \setminus \overline{B}_m$ as the disjoint union of $\log_2 \frac{n}{m}$ annuli, each with aspect ratio 2, and then the one arm probability is bounded by the probability of all successes in $\log_2 \frac{n}{m}$ i.i.d. trials.

4. PROOF OF THEOREM 2.4

Denote $C_{a,b}$ (and $D_{a,b}$) the event that there exists a p_c -open circuit (and p_c -closed dual surface, respectively) in $\overline{B}_b \setminus \overline{B}_a$ that surrounds the origin. By using Corollary 3.2, it is

easy to see that we can choose two alternating sequences $\{n_i\}, \{m_i\}$, such that for each i , $2n_i < m_i < n_{i+1}$, and

$$\begin{aligned} \mathbf{P}_{p_c}[C_{n_i, 2n_i}] &\geq c_0, \\ \mathbf{P}_{p_c}[D_{2n_i, m_i}] &\geq 1 - c_0/2. \end{aligned}$$

We use the total order on circuits defined before the proof of Theorem 3.8. Given $\omega \in C_{n_i, 2n_i}$, we define $\Gamma_{\min}^{(i)}(\omega)$ to be the minimal p_c -open circuit in $\overline{B}_{2n_i} \setminus \overline{B}_{n_i}$ that surrounds the origin. We will omit the superscript i when it is clear from the context.

For $x \in \mathbb{S}_k$ and $n \in \mathbb{N}$ (with x in \overline{B}_n), we denote by \mathcal{I}_x^n the invasion cluster starting at x , and stopped when it first reaches any vertex in $\partial \overline{B}_n$. Let $\mathcal{B}_x^{m_i} = \{\omega : \Gamma_{\min}^{(i)}(\omega) \subset \mathcal{I}_x^{m_i}(\omega)\}$. For any Borel measurable set $A \subset [0, 1]^{\overline{B}_{m_i-1}}$, denote $\mathcal{Y}_A^i = C_{n_i, 2n_i} \cap D_{2n_i, m_i} \cap A$. The following lemma will be proved in Section 4.1.

Lemma 4.1 (Gluing lemma for invasion). *Fix $x \in \mathbb{S}_k$. Take i_0 such that $x \in \overline{B}_{m_{i_0}}$. Then for any $i > i_0$, and any Borel measurable set $A \subset [0, 1]^{\overline{B}_{m_i-1}}$, there exist $C_1, C_2, C_3 < \infty$, such that*

$$\mathbb{P}[(\mathcal{B}_0^{m_i})^c, \mathcal{B}_x^{m_i}, \mathcal{Y}_A^i] \leq C_1 \mathbb{P}[\mathcal{B}_0^{m_i}, \mathcal{B}_x^{m_i}, \mathcal{Y}_A^i], \quad (39)$$

$$\mathbb{P}[\mathcal{B}_0^{m_i}, (\mathcal{B}_x^{m_i})^c, \mathcal{Y}_A^i] \leq C_2 \mathbb{P}[\mathcal{B}_0^{m_i}, \mathcal{B}_x^{m_i}, \mathcal{Y}_A^i], \quad (40)$$

$$\mathbb{P}[(\mathcal{B}_0^{m_i})^c, (\mathcal{B}_x^{m_i})^c, \mathcal{Y}_A^i] \leq C_3 \mathbb{P}[\mathcal{B}_0^{m_i}, \mathcal{B}_x^{m_i}, \mathcal{Y}_A^i]. \quad (41)$$

As a consequence, there exist $c_1, c_2 > 0$, such that

$$\mathbb{P}[\mathcal{B}_0^{m_i}, \mathcal{B}_x^{m_i}, \mathcal{Y}_A^i] \geq c_2 \mathbb{P}[\mathcal{Y}_A^i] \geq c_1 \mathbb{P}[A].$$

Assuming the lemma, we now complete the proof of Theorem 2.4. Denote $\mathcal{Z}^i = \mathcal{B}_0^{m_i} \cap \mathcal{B}_x^{m_i} \cap C_{n_i, 2n_i}$. Since \mathcal{Z}^{i-1} is measurable with respect to the state of edges in $\overline{B}_{m_{i-1}}$, for all i sufficiently large,

$$\mathbb{P}[\mathcal{Z}^i | (\mathcal{Z}^{i_0})^c, (\mathcal{Z}^{i_0+1})^c, \dots, (\mathcal{Z}^{i-1})^c] \geq c_1.$$

It then follows by comparison to a sequence of i.i.d trials with success probability c_1 that

$$\mathbb{P}(\mathcal{B}_0^{m_i}, \mathcal{B}_x^{m_i}, C_{n_i, 2n_i} \text{ i.o.}) = 1.$$

Finally, notice that since $\mathcal{B}_0^{m_i} \subset \{\omega : \Gamma_{\min}^{(i)}(\omega) \subset \mathcal{I}_0(\omega)\}$, $\mathcal{B}_x^{m_i} \subset \{\omega : \Gamma_{\min}^{(i)}(\omega) \subset \mathcal{I}_x(\omega)\}$, we can conclude that

$$\mathbb{P}[\Gamma_{\min}^{(i)} \subset \mathcal{I}_0, \Gamma_{\min}^{(i)} \subset \mathcal{I}_x, C_{n_i, 2n_i} \text{ i.o.}] = 1.$$

In particular, $\mathcal{I}_0 \cap \mathcal{I}_x \neq \emptyset$ a.s.

4.1. Proof of Lemma 4.1. In order to prove Lemma 4.1, we start with the following extension of the combinatorial Lemma 7 of [DST15a]. It concerns maps Φ on the edge labels $\omega = \{\omega(e), e \in E\}$ such that Φ decreases (respectively, increases) finitely many $\omega(e)$'s in an affine way in order to make those edges open (respectively, closed).

Lemma 4.2. Consider $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$, $a, b \in (0, 1)$, and a measurable map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$. If for any $\omega' \in \Phi(\mathcal{A})$, there exists $S(\omega') \subset E$ with less than or equal to s edges, such that

$$\Phi^{-1}(\omega') \subset \left\{ \omega : \omega|_{S^c} = \omega'|_{S^c} \right\} \cap \left[\bigcup_{L \subset S} \left(\left\{ \omega : \omega|_L = \frac{1}{a} \omega'|_L \right\} \cap \left\{ \omega : \omega|_{S \setminus L} = \frac{\omega' - b}{1 - b} |_{S \setminus L} \right\} \right) \right],$$

then $\mathbb{P}[\mathcal{A}] \leq \left(\frac{2}{a \wedge (1-b)} \right)^s \mathbb{P}[\mathcal{B}]$.

Roughly speaking, this lemma says that if one can obtain \mathcal{B} by modifying a small number of edges in \mathcal{A} , in a way that given any element in \mathcal{B} , the number of its pre-images is bounded, then $\mathbb{P}[\mathcal{A}]$ can be bounded from above by a constant times $\mathbb{P}[\mathcal{B}]$.

Remarks.

1. An equivalent way of stating the hypotheses on Φ is that Φ leaves all but at most s of the $\omega(e)$'s unchanged, with the others either lowered (by $\omega(e) \mapsto a\omega(e)$), or raised (by $\omega(e) \mapsto b + (1-b)\omega(e)$) and the set S of changed edges is uniquely determined by $\omega' = \Phi(\omega)$.
2. In Lemma 4.2, all the edges in S have their edge labels either decreased or increased. Although it is not needed in this paper, we note that the lemma can be extended to allow for some of the edges in S to be unchanged.

Proof of Lemma 4.2. First observe that for any $\omega' \in \Phi(\mathcal{A})$, $\text{Card}(\Phi^{-1}(\omega')) \leq 2^{|S(\omega')|} \leq 2^s$. Therefore one can take a disjoint partition $\{A_i\}_{i=1}^{2^s}$ (some of which may be empty) of \mathcal{A} , such that $\Phi|_{A_i}$ is a bijection. Indeed, there is an $L_i(\omega)$ for $\omega \in A_i$, such that

$$\Phi|_{A_i}(\omega)(e) = \begin{cases} \omega(e) & \text{if } e \notin S(\Phi(\omega)) \\ b + (1-b)\omega(e) & e \in S(\Phi(\omega)) \setminus L_i \\ a\omega(e) & e \in L_i \end{cases}.$$

Then one can bound its Jacobian $J_i(\omega)$ from below by

$$J_i(\omega) \geq a^{\text{Card}(L_i)} (1-b)^{\text{Card}(S(\Phi(\omega)) \setminus L_i)} \geq (a \wedge (1-b))^s.$$

Therefore,

$$\begin{aligned} \mathbb{P}[\mathcal{B}] &\geq \int_{\Phi(A_i)} d\omega' = \int_{A_i} J_i(\omega) d\omega \\ &\geq \int_{A_i} (a \wedge (1-b))^s d\omega = (a \wedge (1-b))^s \mathbb{P}[A_i]. \end{aligned}$$

Summing over i , we obtain

$$2^s \mathbb{P}[\mathcal{B}] \geq (a \wedge (1-b))^s \mathbb{P}[\mathcal{A}].$$

□

Proof of Lemma 4.1. We now prove (39) by explicitly constructing a map $\Phi : (\mathcal{B}_0^{m_i})^c \cap \mathcal{B}_x^{m_i} \cap \mathcal{Y}_A^i \rightarrow \mathcal{B}_0^{m_i} \cap \mathcal{B}_x^{m_i} \cap \mathcal{Y}_A^i$ that satisfies the hypothesis of Lemma 4.2. The proof of (40) follows from the same argument, and the proof of (41) will be described at the end of this proof. Given $\omega \in (\mathcal{B}_0^{m_i})^c \cap \mathcal{B}_x^{m_i} \cap C_{n_i, 2n_i} \cap D_{2n_i, m_i}$ with $i > i_0$, let $R(\omega)$ denote

the connected component containing 0 in $\{w \in \mathbb{S}_k : \text{dist}(\bar{w}, \Gamma_{\min}(\omega)) \geq 1\}$. In particular, $\partial R \subset \{w \in \mathbb{S}_k : \text{dist}(\bar{w}, \Gamma_{\min}(\omega)) = 1\}$. Let $\tau_i = \min\{j : \mathcal{I}_0[j] \in \partial R\}$ and $z(\omega) = \mathcal{I}_0[\tau_i]$ be the first landing point of \mathcal{I}_0 on ∂R . By definition, there exists $z' \in \Gamma_{\min}$ such that $\text{dist}(\bar{z}, \bar{z}') = 1$. If there exists more than one z' satisfy $\text{dist}(\bar{z}, \bar{z}') = 1$, choose the minimal one.

Recall that $\mathcal{C}_{p_c}(\Gamma_{\min}(\omega))$ denotes the p_c -open cluster containing $\Gamma_{\min}(\omega)$. Notice that $\omega \in (\mathcal{B}_0^{m_i})^c$ implies

$$z(\omega) \notin \mathcal{C}_{p_c}(\Gamma_{\min}(\omega)). \quad (42)$$

Otherwise, by the observation in Section 2.2, we would have $\Gamma_{\min}(\omega) \subset \mathcal{I}_0^{m_i}(\omega)$. To complete the proof, we will use $B_1^\#(z)$ for $z \in \mathbb{S}_k$ to denote $z + \{(0, 0, 0), (1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0)\}$; as usual, $\bar{B}_1^\#(z)$ denotes the cylinder in \mathbb{S}_k generated by the five-point set $B_1^\#(z)$. There exists a self avoiding path Γ_z in $\bar{B}_1^\#(z')$ connecting $z(\omega)$ to $\Gamma_{\min}(\omega)$ without touching any other vertices in $\Gamma_{\min}(\omega) \cup \partial \bar{B}_1^\#(z')$. In particular one can construct Γ_z by taking one edge from z to \bar{z}' and then move in \bar{z}' until reaching the first vertex in Γ_{\min} .

We now construct $\omega' = \Phi(\omega)$ as follows.

- (1) Open all the edges in Γ_z (that is, take $\omega(e) \mapsto p_c \omega(e)$).
- (2) If $\mathcal{I}_x^{m_i}$ touches $\bar{B}_1^\#(z')$, then proceed as follows; otherwise go to Step 3. Define $\tau_i^x = \min\{j : \mathcal{I}_x[j] \in \partial \bar{B}_1^\#(z')\}$ and $w = \mathcal{I}_x[\tau_i]$ to be the first vertex in $\partial \bar{B}_1^\#(z')$ reached by the invasion cluster starting from x . If $w \in \mathcal{C}_{p_c}(z)$, go to Step 3. If $w \notin \mathcal{C}_{p_c}(z)$, then there exists a self avoiding path Γ_w in $\bar{B}_1^\#(z')$ connecting $w(\omega)$ to $\Gamma_{\min}(\omega) \cup \Gamma_z$ without touching any other vertices in $\Gamma_{\min}(\omega) \cup \Gamma_z \cup \partial \bar{B}_1^\#(z')$. Open all the edges in Γ_w .
- (3) Close all the edges in $\bar{B}_1^\#(z')$ (that is, map $\omega(e) \mapsto b + (1 - b)\omega(e)$ with $b > p_c$) except for the edges of $\Gamma_{\min}(\omega) \cup \Gamma_z \cup \Gamma_w$.

By construction, $\omega' \in \mathcal{B}_0^{m_i} \cap \mathcal{B}_x^{m_i} \cap \mathcal{Y}_A^i$. To see this, note that when D_{2n_i, m_i} occurs, $\mathcal{I}_0^{m_i}$ (or $\mathcal{I}_x^{m_i}$) touching any vertex $v \in \bar{B}_{2n_i}(0)$ implies it also contains all of $\mathcal{C}_{p_c}(v)$. To apply Lemma 4.2, we need to bound the number of pre-images of ω' by 2^s for some s . For this, we first note the important feature that

$$\Gamma_{\min}(\omega') = \Gamma_{\min}(\omega). \quad (43)$$

This will help show that the set S of changed edges is uniquely determined by ω' .

Indeed, the construction will not create any new p_c -open circuits. If the construction skips Step 2, then any new p_c -open circuit would contain a subset of Γ_z , and then it would contain all of Γ_z because of Step 3. Therefore if the construction created some new p_c -open circuit, we would have $z(\omega) \in \mathcal{C}_{p_c}(\Gamma_{\min}(\omega))$, which would contradict (42). If the construction uses Step 2, by the same argument, we would have either $z(\omega) \in \mathcal{C}_{p_c}(\Gamma_{\min}(\omega))$, or $w(\omega) \in \mathcal{C}_{p_c}(\Gamma_{\min}(\omega))$, or $w(\omega) \in \mathcal{C}_{p_c}(z(\omega))$, any of which would lead to a contradiction.

Now, given $\omega' = \Phi(\omega)$, one can determine $S(\omega) = S(\omega')$ as follows.

- (1) Thanks to (43), $R(\omega') = R(\omega)$, and $\omega|_{R(\omega)} = \omega'|_{R(\omega')}$ (where $\omega|_R$ here means the set of edge labels with both vertices in R). This implies $z(\omega) = z(\omega')$. Therefore one can explore $\mathcal{I}_0^{m_i}(\omega')$ until it contains z , without any change from $\mathcal{I}_0^{m_i}(\omega)$.
- (2) $z'(\omega') = z'(\omega)$. In fact, $z' \in \Gamma_{\min}(\omega') = \Gamma_{\min}(\omega)$ is uniquely characterized by $\text{dist}(\bar{z}, z') = 1$ and the minimality of z' .
- (3) Taking $S(\omega') = \bar{B}_1^\#(z') \setminus \Gamma_{\min}(\omega')$, we see that $\omega|_{S^c} = \omega'|_{S^c}$, and that the map Φ satisfies the conditions of Lemma 4.2 with s equal to the number of edges in $\bar{B}_1^\#$. Applying Lemma 4.2 we obtain that

$$\begin{aligned} & \mathbb{P}[(\mathcal{B}_0^{m_i})^c, \mathcal{B}_x^{m_i}, \mathcal{Y}_A^i] \\ & \leq \left(\frac{2}{p_c \wedge (1-b)} \right)^s \mathbb{P}[\mathcal{B}_0^{m_i}, \mathcal{B}_x^{m_i}, \mathcal{Y}_A^i], \end{aligned}$$

which concludes the proof of (39) (and similarly (40)).

Finally, to prove (41), we note that a map

$$\Phi : (\mathcal{B}_0^{m_i})^c \cap (\mathcal{B}_x^{m_i})^c \cap \mathcal{Y}_A^i \rightarrow (\mathcal{B}_0^{m_i} \cap \mathcal{B}_x^{m_i} \cap \mathcal{Y}_A^i) \cup (\mathcal{B}_0^{m_i} \cap (\mathcal{B}_x^{m_i})^c \cap \mathcal{Y}_A^i)$$

can be constructed in essentially the same way as above (in fact, one can skip Step 2 when constructing ω'). Lemma 4.2 then implies

$$\mathbb{P}[(\mathcal{B}_0^{m_i})^c, (\mathcal{B}_x^{m_i})^c, \mathcal{Y}_A^i] \leq C_4 \mathbb{P}[\mathcal{B}_0^{m_i}, \mathcal{B}_x^{m_i}, \mathcal{Y}_A^i] + C_4 \mathbb{P}[\mathcal{B}_0^{m_i}, (\mathcal{B}_x^{m_i})^c, \mathcal{Y}_A^i],$$

for some $C_4 < \infty$. Together with (40) we obtain (41) with $C_3 = C_4(1 + C_2)$. \square

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